Unit-2: Time Response Analysis Of Standard Test Signals

Time Response

Introduction:

- After deriving a mathematical model of a system, the system performance analysis can be done in Various methods.
- In analyzing and designing Control systems, a basis of Comparision performance of Various Control systems should be made.
- This basis may be set up by Specifying particular test input signals and by comparing the responses of Various systems to these signals.
- The system stability, system accuracy and Complete evaluation are always based on the time response analysis and the corresponding results.
- Next important step after a mathematical model of a system ~~to~~ an is obtained is to analyze the system's performance.
- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

$$\boxed{\begin{aligned} G(s) &= \frac{C(s)}{R(s)} \\ g(t) &= \frac{c(t)}{R(t)} \end{aligned}}$$

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.
- Usually, the input signals to Control systems are not Known fully ahead of time.

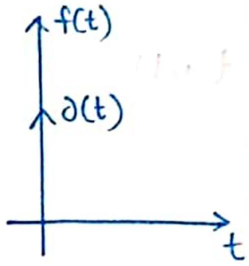
- For example, in a radar tracking system, the position and the speed of the target to be tracked may vary in a random fashion.
- It is therefore difficult to express the actual input signals mathematically by simple equations.

$$\boxed{\begin{aligned} c(s) &= G(s)\,R(s) \\ c(t) &= L^{-1}\{G(s)\,R(s)\} \\ &= \int_0^t g(\tau)\, r(t-\tau)\, d\tau \end{aligned}}$$

τ - Tow (Torque)

- The characteristics of actual input signals are a sudden shock (impulse) a sudden change (step) a constant velocity (ramp) and constant acceleration (parabolic).
- The dynamic behaviour of a system is therefore judged and compared under application of standard test signals - an impulse, a step, a constant velocity and constant acceleration.
- Another standard signal of great importance is a sinusoidal signal.
- Normally use the standard input signals to identify the characteristics of system's response.
  - step function
  - Ramp function
  - Impulse function
  - parabolic function
  - sinusoidal function

Impulse Signal

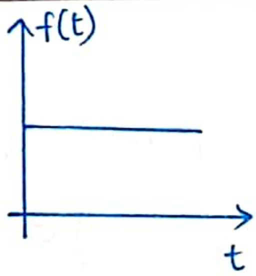
Input	function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0^- < t < 0$ $= 0$ elsewhere $\int_{0^-}^{0^+} \delta(t)\, dt = 1$		Transient response modeling

$$\left.\begin{aligned} \delta(t) &= \infty \quad t=0 \\ &= 0 \quad t \neq 0 \end{aligned}\right\} \text{Ideal impulse input}$$

$$\left.\begin{aligned} \delta(t) &= 1 \quad t=0 \\ &= 0 \quad t \neq 0 \end{aligned}\right\} \text{Unit Impulse input}$$

$$\left.\begin{aligned} \delta(t) &= A \quad t=0 \\ &= 0 \quad t \neq 0 \end{aligned}\right\} \text{Impulse input.}$$

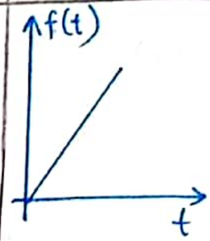
Step signal

Input	function	Description	Sketch	Use
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response steady-state error

$f(t) = u(t) \longrightarrow$ Unit step input

$f(t) = A \cdot u(t) \longrightarrow$ step input

Ramp Signal

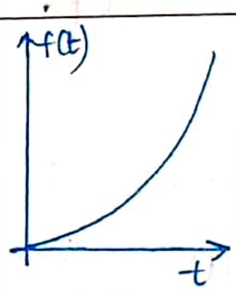
Input	function	Description	Sketch	Use
Ramp	$t u(t)$	$t u(t) = t$ for $t > 0$ $= 0$ elsewhere		Steady-state error.

$$f(t) = t \cdot u(t)$$

$$\left.\begin{aligned} f(t) &= t & t &\geq 0 \\ &= 0 & t &< 0 \end{aligned}\right\}$$ Unit ramp input

$$\left.\begin{aligned} f(t) &= A \cdot t & t &\geq 0 \\ &= 0 & t &< 0 \end{aligned}\right\}$$ ramp input

Parabolic Signal

Input	function	Description	Sketch	Use
Parabola	$\frac{1}{2}t^2 u(t)$	$\frac{1}{2}t^2 u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error

$f(t) = \frac{1}{2}t^2 u(t) \longrightarrow$ Unit parabolic input

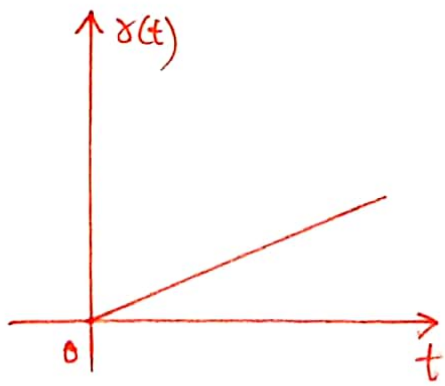
$f(t) = \frac{A}{2}t^2 \cdot u(t) \longrightarrow$ Parabolic input.

Standard Test signals

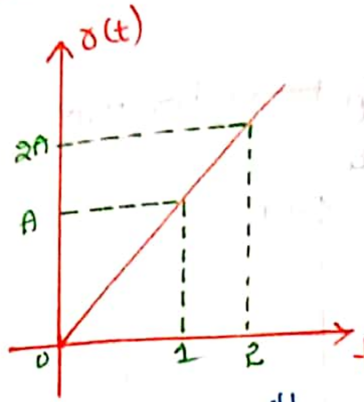
Ramp Signal :

- The ramp signal imitate the constant velocity characteristics of actual input signal.

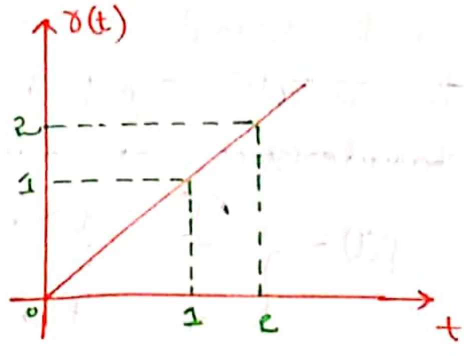
$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- If $A = 1$, the ramp signal is called Unit ramp signal.



Ramp signal with slope A

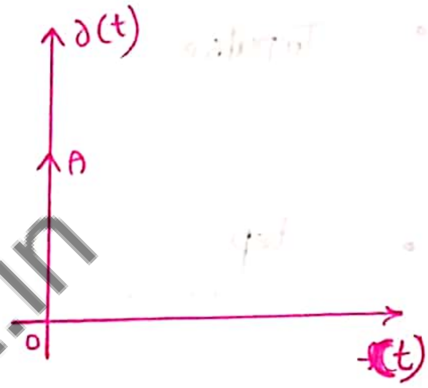


Unit ramp signal

Impulse Signal :

- The impulse signal imitate the sudden shock characteristics of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

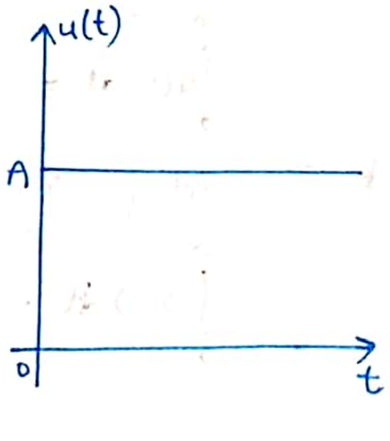


- If A=1, the impulse signal is called Unit impulse signal

Step Signal :

- The step signal imitate the sudden change characteristics of actual input signal

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

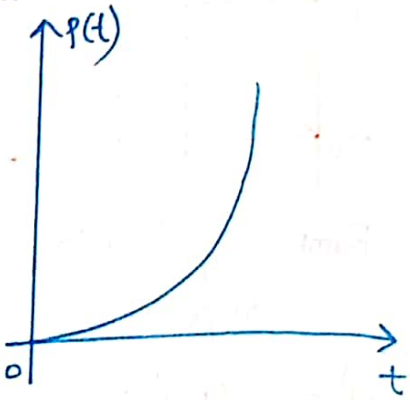


- If A=1, the step signal is called Unit step signal.

Parabolic Signal

- The parabolic signal imitate the constant acceleration characteristics of actual input signal.

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- If A=1, the parabolic signal is called Unit parabolic signal.

Relation between standard Test Signals.

- Impulse $\quad \delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$

- Step $\quad u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$

- Ramp $\quad r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$

- Parabolic $\quad p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$

* On integrating impulse we get step

$$\int_0^\infty \delta(t)\, d(t) = \delta(t) \int_0^\infty = A = u(t)$$

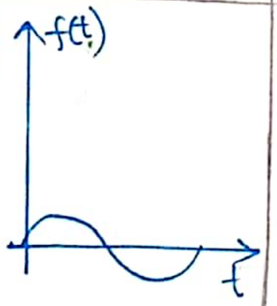
* On integrating step we get ramp

$$\int_0^\infty u(t)\, dt = u(t) \int_0^\infty = At = r(t)$$

* On integrating ramp we get parabolic

$$\int_0^\infty r(t)\, dt = p(t) \Big|_0^\infty = \frac{At^2}{2}$$

Sinusoidal Signal

Input	function	Description	Sketch	Use
Sinusoid	$\sin\omega t$	$f(t) = A\sin\omega t$; $\omega = \frac{2\pi}{T}$		Transient response Modeling Steady-State error

Test signals.

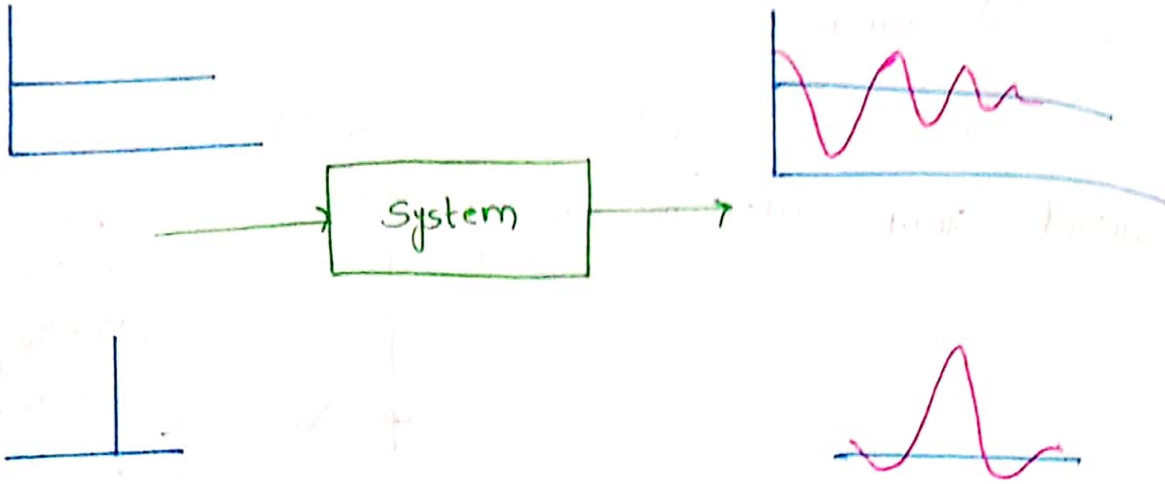
Input	$r(t)$	$R(s)$
Step Input	A	A/s
Ramp Input	At	A/s^2
Parabolic Input	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Impulse Input	$\delta(t)$	1

Test signals

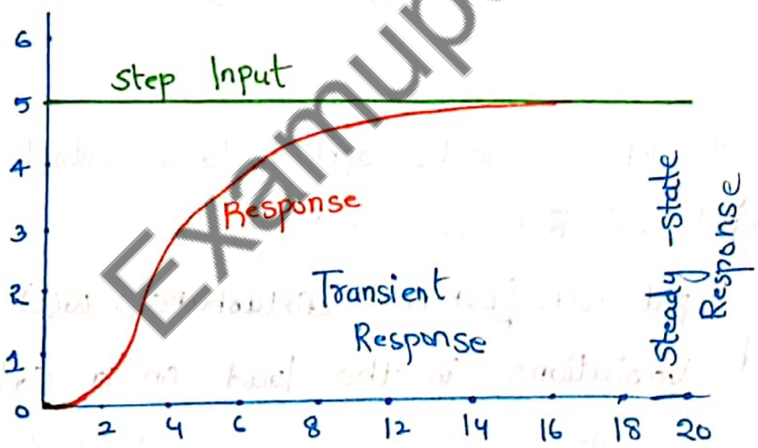
- Two types of inputs can be applied to a control system. Command input (or) Reference input $y_r(t)$.
- Disturbance input $w(t)$ [External disturbances $w(t)$ are typically uncontrolled variations in the load on a control system]
- In systems controlling mechanical motions, load disturbances may represent forces.
- In voltage regulating systems, variations in electrical load are a major source of disturbances.

Time Response of Control Systems

- Time response of a dynamic system is response to an input expressed as a function of time



- The time response of any system has two Components.
  - Transient response.
  - Steady-state response.
- When the response of the system is changed form rest (or) equilibrium it takes some time to settle down...
- Transient response is the response of a system from rest (or) equilibrium to steady state.



- The response of the system after the transient response is called steady state response.

Transient Response

- The transient response is defined as the part of the time response that goes to zero as time becomes very large.
- Thus $y_t(t)$ has the property $\lim_{t \to \infty} y_t(t) = 0$
- The time required to achieve the final value is called transient period.

- The transient response may be exponential (or) oscillatory in nature.
- Output response Consists of the sum of forced response (from the input) and natural response (from the nature of the system).
- The transient response is the change in output response from the beginning of the response to the final state of the response. and the steady state response is the output response as time is approaching infinity (or no changes at the output).

Steady state Response:

- The steady state response is the part of the total response that remains after the transient has died out.
- For a position Control system, the steady state response when Compared to with the desired reference position gives an indication of the final accuracy of the system.
- If the steady state response of the output does not agree with the desired reference exactly, the system is said to have steady error.

Time Response of Control systems

- Transient response dependents upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final Value theorem

Time Response of First order system

Introduction :

- The first order system has only one pole

$$\boxed{\frac{C(s)}{R(s)} = \frac{K}{Ts+1}}$$

Where K is the DC gain and
T is the time constant of the system.

- Time Constants is a measure of how quickly a 1st order system responds to a unit step input.
- DC Gain of the system is ratio between the input signal and the steady state value of output.
- For the first order system given

$$\boxed{G(s) = \frac{10}{3s+1}}$$

DC gain is 10 and time constant is 3 seconds

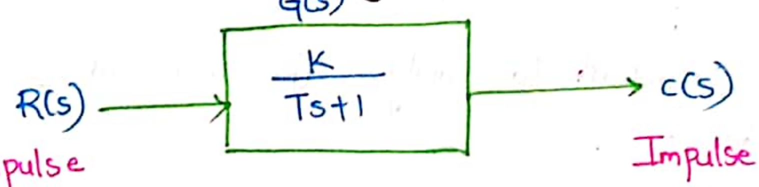
- And for following system

$$\boxed{G(s) = \frac{3}{s+5} = \frac{\frac{3}{5}}{\left(\frac{1}{5}\right)s+1}}$$

Dc Gain of the system is $\frac{3}{5}$ and time constant is $\frac{1}{5}$ sec

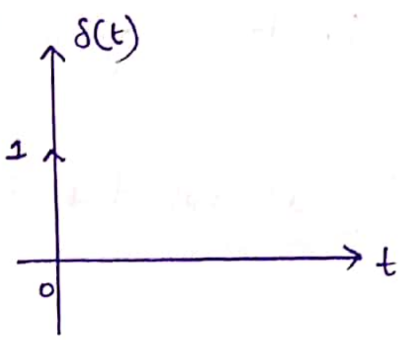
Impulse Response of 1st order system.

- Consider the following 1st order system



Impulse input

Impulse response
output of impulse input
response to impulse input



$R(s) = 1 \longrightarrow$ impulse input

$$c(s) = G(s)\, R(s)$$

$$\boxed{c(s) = G(s)}$$

Taking inverse laplace

$$L^{-1}\{c(s)\} = L^{-1}\{G(s)\}$$

$$\boxed{c(t) = g(t)}$$

$g(t) \Leftrightarrow G(s) \longrightarrow$ Impulse Response

$$c(s) = \frac{K}{Ts+1}$$

Rearrange above equation as

$$\boxed{c(s) = \frac{\frac{K}{T}}{s + \frac{1}{T}}}$$

In order to represent the response of the system in time domain we need to compute inverse laplace transform of the above equation.

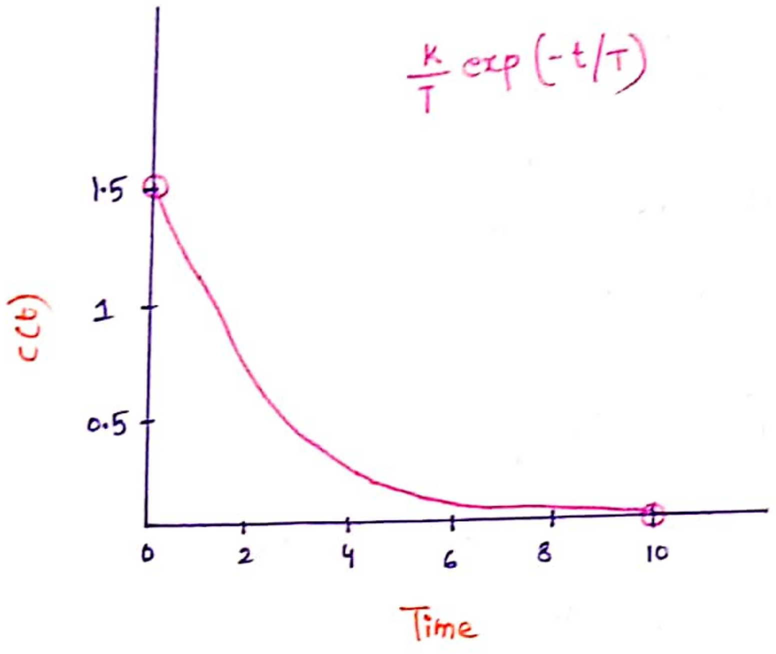
$$L^{-1}\left\{\frac{A}{s+a}\right\} = Ae^{-at}$$

$$L^{-1}\{c(s)\} = \frac{K}{T} e^{-t/T}$$

$$\therefore \boxed{c(t) = \frac{K}{T} e^{-t/T}}$$

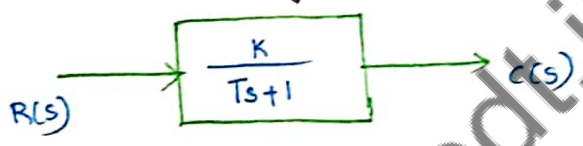
If $K=3$ and $T=2s$ then

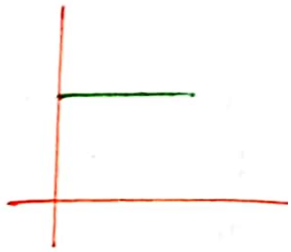
$$c(t) = \frac{K}{T} e^{-t/T}$$



Step Response of 1st order system

- Consider the following 1st order system





$$R(s) = U(s) = \frac{1}{s}$$

$$\boxed{C(s) = \frac{K}{s(Ts+1)}}$$

In order to find out the inverse laplace of the above equation need to break it into partial expression.

$$\boxed{\begin{aligned} C(s) &= G(s)\, R(s) \\ C(s) &= \frac{K}{Ts+1}\left(\frac{1}{s}\right) \end{aligned}}$$

$$C(s) = \frac{A}{s} - \frac{B}{Ts+1}$$

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

$$\boxed{C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)}$$

Taking inverse laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

Where $u(t) = 1$

$$c(t) = K\left(1 - e^{-t/T}\right)$$

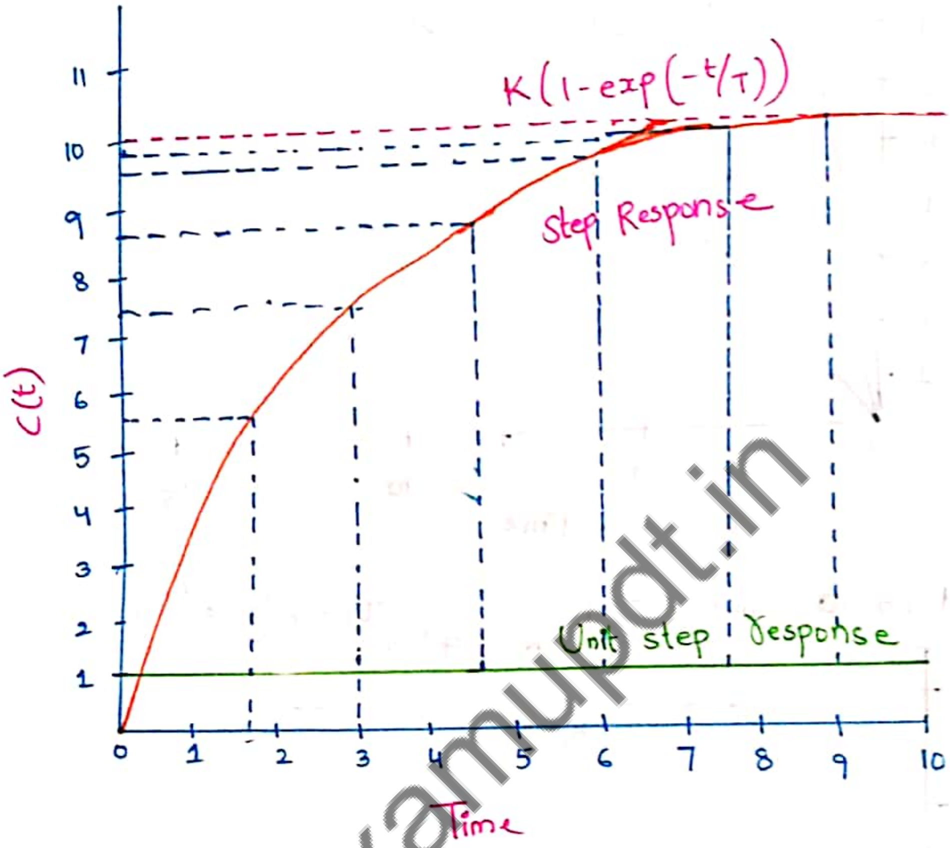
When $t = T$

$$c(t) = K(1 - e^{-1})$$

$$c(t) = 0.632K$$

If $k = 10$ and $T = 1.5s$ then

$$c(t) = K(1 - e^{-t/T})$$



$$\text{DC Gain } (K) = \frac{\text{steady state output}}{\text{Input}} = \frac{10}{1}$$

$$\boxed{K = 10}$$

$t = (0)$ $\quad c(t) = 0$

$t = T$ $\quad c(t) = 0.632k$

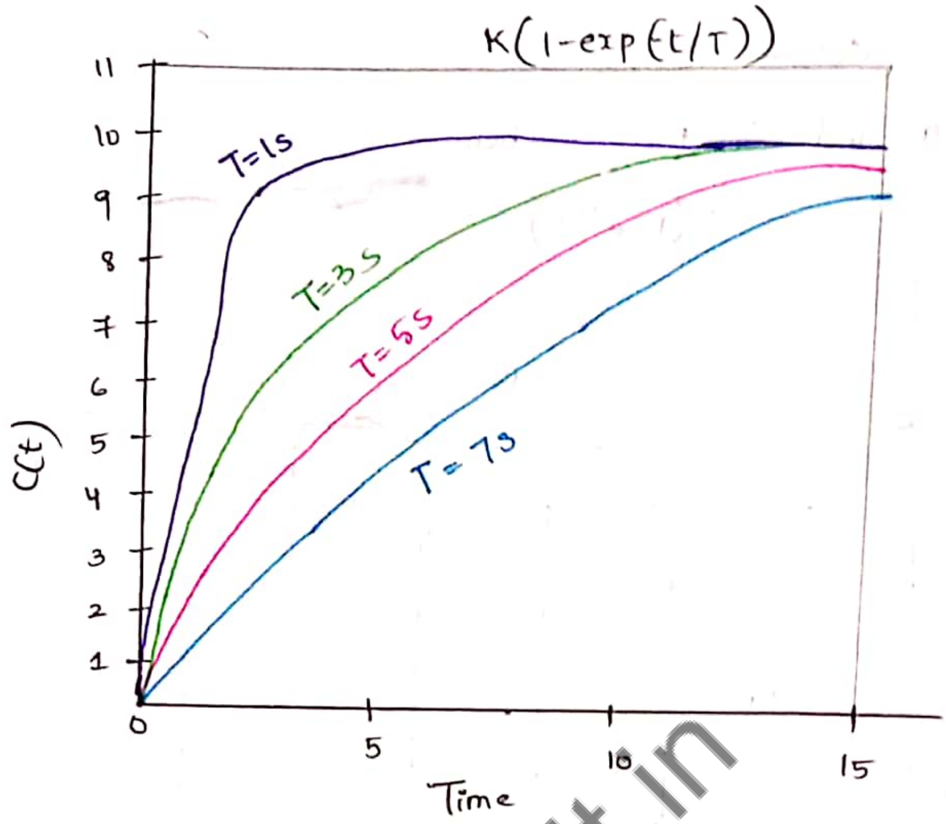
$t = 2T$ $\quad c(t) = 0.864k$

$t = 3T$ $\quad c(t) = 0.953k$

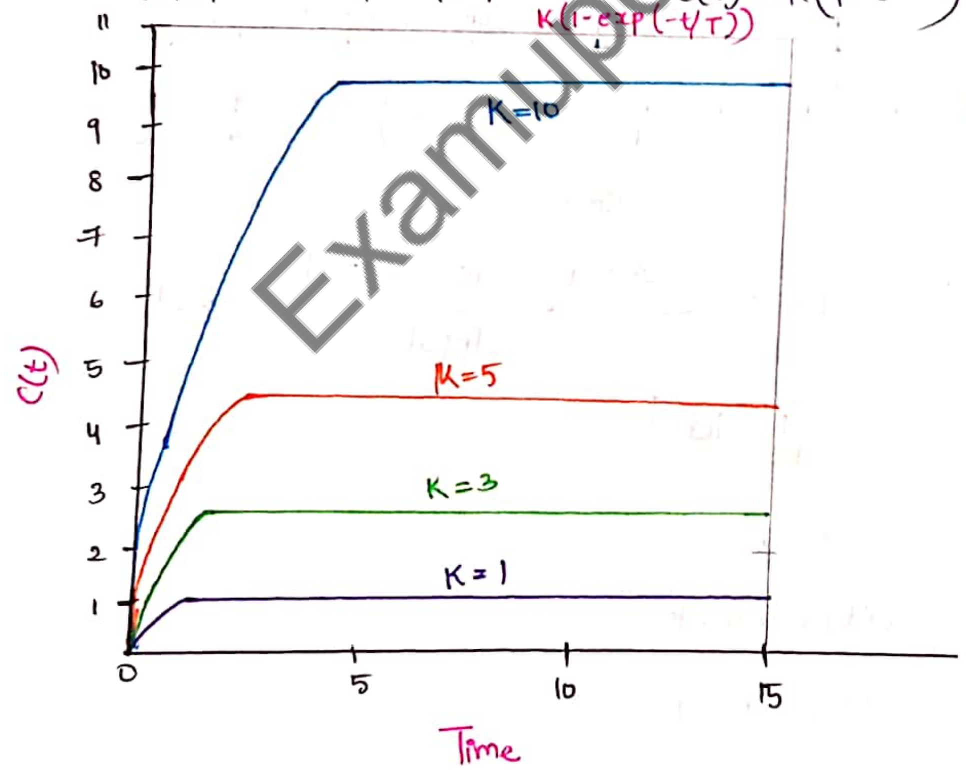
$t = 4T$ $\quad c(t) = 0.982k$

$t = 5T$ $\quad c(t) = 0.99k$

• If $K=10$ and $T=1,3,5,7$ $\quad c(t) = K(1-e^{-t/T})$



• If $K=1,3,5,10$ and $T=1$ $\quad c(t) = K(1-e^{-t/T})$



Relation Between Step and Impulse Response

- The step response of the first order system is

$$c(t) = K(1-e^{-t/T}) = K - Ke^{-t/T}$$

Differentiating c(t) with respect to t yields.

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T} e^{-t/T} \quad \text{(Impulse response)}$$

Example-1:

Impulse response of a 1^{st} order system is given below

$$c(t) = 3e^{-0.5t}$$

Find out. i) Time Constant (T) ii) DC Gain (K)

iii) Transfer function iv) Step response

Ans: • The Laplace Transform of Impulse response of a system is actually the transfer function of the system.

- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

Sir slide

$r(s) = \delta(s)$

$$C(s) = \frac{3}{s+0.5} \times 1 = \frac{3}{s+0.5}\,\delta(s)$$

$$\frac{1}{T} = 0.5 \qquad T = 2s$$

$$\frac{K}{T} = 3 \qquad K = 3T = 6$$

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)\,dt = 3\int e^{-0.5t}\,dt$$

$$c_s(t) = -6e^{-0.5t} + c$$

We can find out c,
if initial condition is known

$$C_s(0) = 0 .$$

$$0 = -6\, e^{-0.5(0)} + c \qquad \boxed{c = 6}$$

$$\boxed{C_s(t) = 6 - 6e^{-0.5t}}$$

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2s+1}$$

Since R(s) is a step input

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{6}{s(2s+1)}$$

Consider $\dfrac{6}{s(2s+1)} = \dfrac{A}{s} + \dfrac{B}{2s+1}$

$$\frac{6}{s(2s+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$\boxed{c(t) = 6 - 6e^{-0.5t}}$$

First order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at $-1/\alpha$ and pole at $-1/T$
- Step response of the system would be:

$$c(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$c(s) = \frac{K}{s} + \frac{K(\alpha - T)}{Ts+1} \qquad \text{(Partial Fractions)}$$

$$\boxed{c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}} \qquad \text{(Inverse laplace)}$$

Fast order System with and without zero (comparision)

With Zero

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

$$c(t) = K\left(1 - e^{-t/T}\right)$$

Without Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

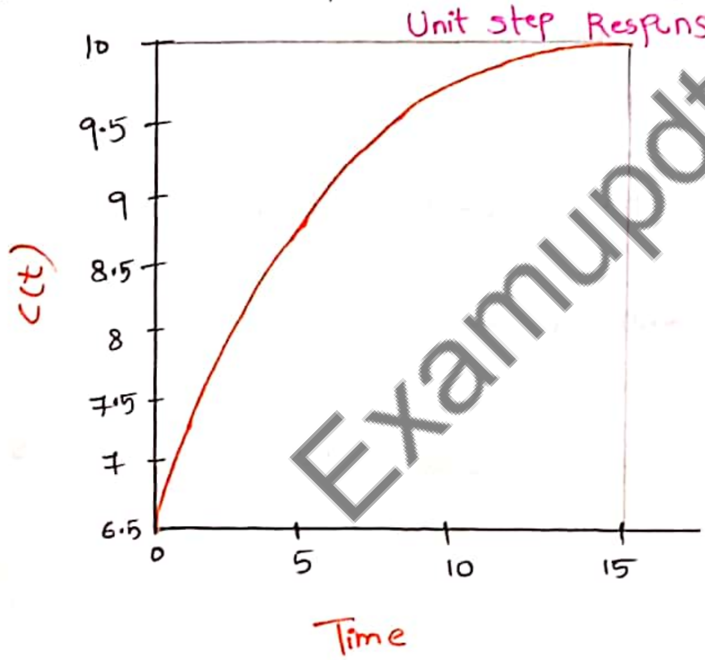
$$c(t) = K + \frac{K}{T}(\alpha - T)\, e^{-t/T}$$

- If $T > \alpha$ the shape of the slep desponse is approximately same (with offset added by zero)

$$c(t) = K + \frac{K}{T}(-n)\, e^{-t/T}$$

$$\boxed{c(t) = K\left(1 - \frac{n}{T} e^{-t/T}\right)}$$

- If $T > \alpha$ the desponse of the system would look like

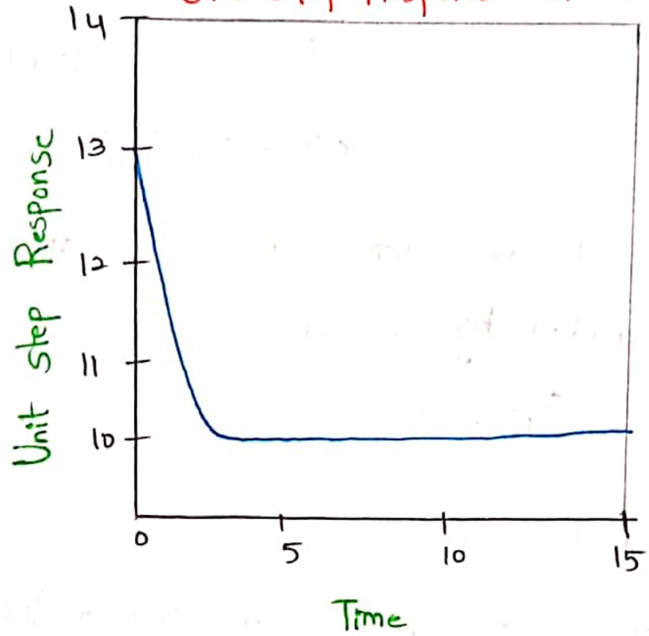


$$\frac{C(s)}{R(s)} = \frac{10(1+2s)}{3s+1}$$

$$\therefore \boxed{c(t) = 10 + \frac{10}{3}(2-3)\, e^{-t/3}}$$

- If $T<\alpha$ the response of the system would look like

Unit step Response of 1st order system with zeros

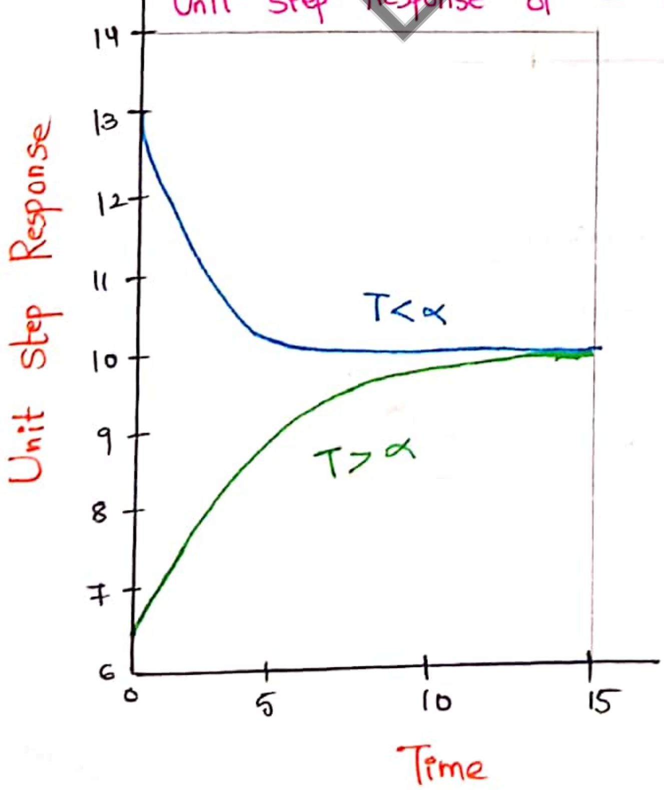


$$\frac{C(s)}{R(s)} = \frac{10(1+2s)}{1.5s+1}$$

$$\boxed{c(t) = 10 + \frac{10}{1.5}(2-1)\, e^{-t/1.5}}$$

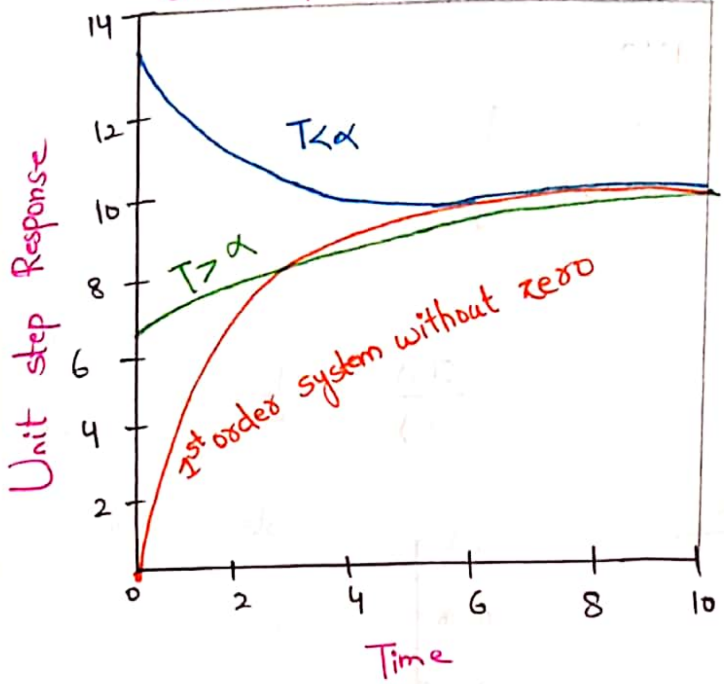
First order System with a Zero

Unit step Response of 1st order systems with zeros

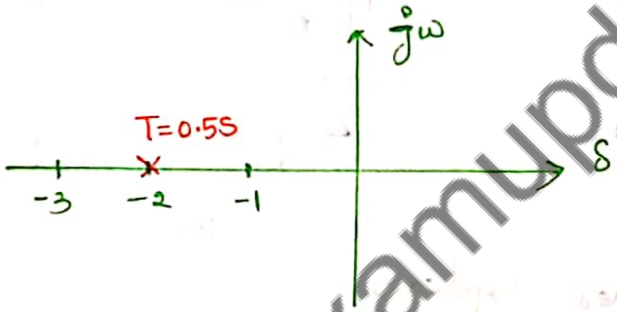


First order System with and without zero.

Unit step Response of 1^{st} order systems.



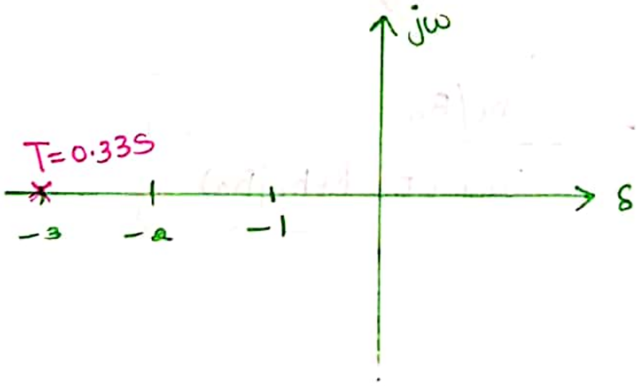
P-Z map and Step Response



$$\frac{C(S)}{R(S)} = \frac{K}{TS+1}$$

$$\frac{C(S)}{R(S)} = \frac{10}{S+2}$$

$$\boxed{\frac{C(S)}{R(S)} = \frac{5}{0.5S+1}}$$



$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

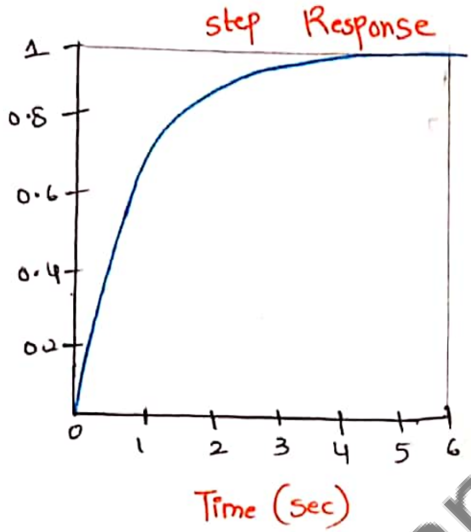
$$\frac{C(s)}{R(s)} = \frac{10}{s+3}$$

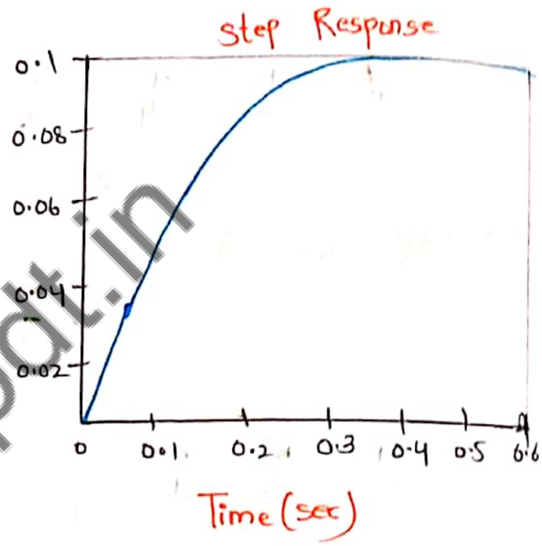
$$\boxed{\frac{C(s)}{R(s)} = \frac{3.3}{0.33s+1}}$$

Comparision.

$$\frac{C(s)}{R(s)} = \frac{1}{s+1}$$

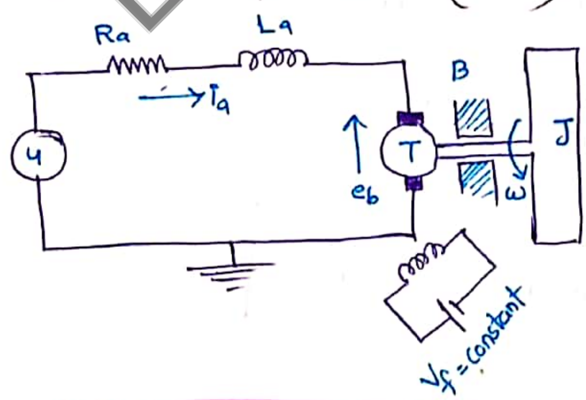
$$\frac{C(s)}{R(s)} = \frac{1}{s+10} = \frac{0.1}{0.1s+1}$$





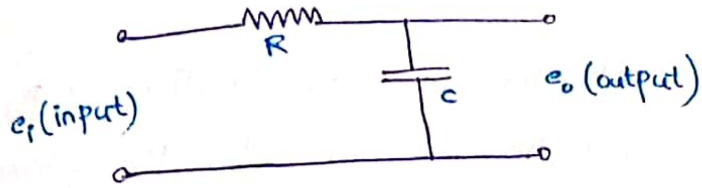
Examples of First order Systems

- Armature Controlled DC motor ($L_a = 0$)



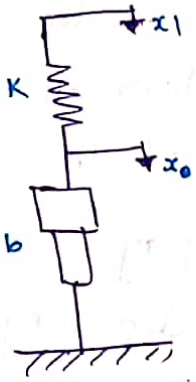
$$\boxed{\frac{\Omega(s)}{U(s)} = \frac{K_t/R_a}{Js + (B + K_t K_b/R_a)}}$$

- Electrical system



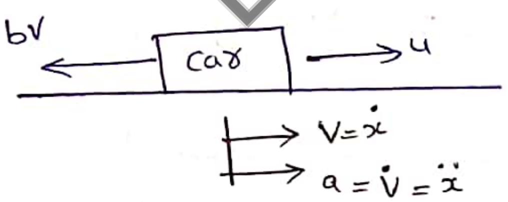
$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{RCs+1}}$$

- Mechanical system



$$\boxed{\frac{X_o(s)}{X_i(s)} = \frac{1}{\frac{b}{K}s+1}}$$

- Cruise Control of Vehicle

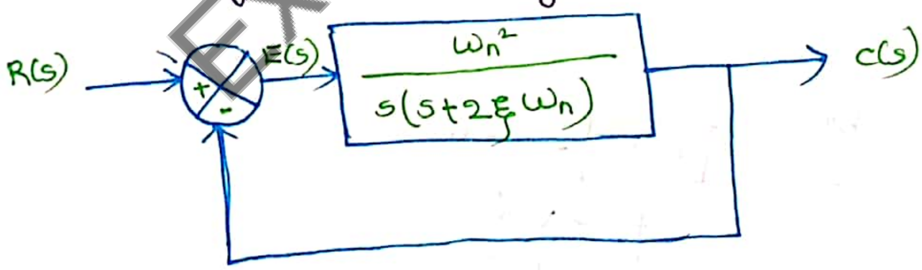


$$\boxed{\frac{V(s)}{U(s)} = \frac{1}{ms+b}}$$

Second Order Systems

Introduction:

- We have already discussed the affect of location of pole and zero on the transient response of 1^{st} order systems
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameters (T, K) simply changes the speed and offset of the response.
- Whereas, changes in the parameters of a second-order system can change the form of the response.
- A second-order system can display characteristics much like a first-order system (or) depending on component values, display damped (or) pure oscillations for its transient response.
- A general second-order system (without zeros) is characterized by the following transfer function.



$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}$$

Where ξ (or) ζ – Damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output

ω_n – Undamped natural frequency of the 2^{nd} order system, which is the frequency of oscillation of the system without damping.

Example - 1 :

Determine the Undamped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$$

sol: Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$s^2+2\xi\omega_n s+\omega_n^2 = s^2+2s+4$$

$$\omega_n^2 = 4$$

$$\boxed{\omega_n = 2}$$

$$2\xi\omega_n s = 2s$$

$$2\xi\omega_n = 2$$

$$\boxed{\xi = 0.5}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

- The closed-loop poles of the system are

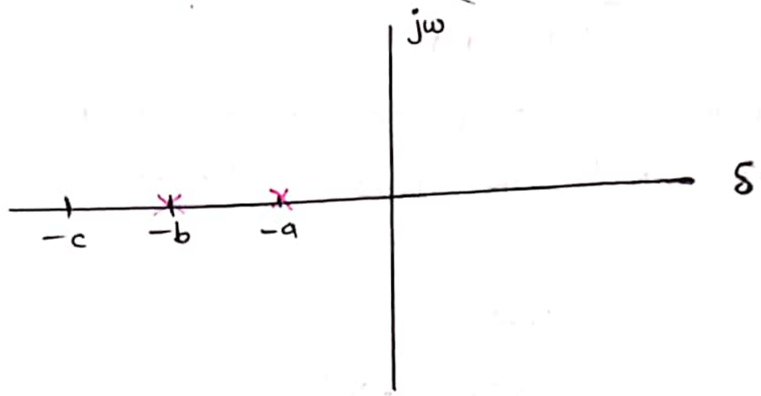
$$-\omega_n\xi + \omega_n\sqrt{\xi^2-1}$$

$$-\omega_n\xi - \omega_n\sqrt{\xi^2-1}$$

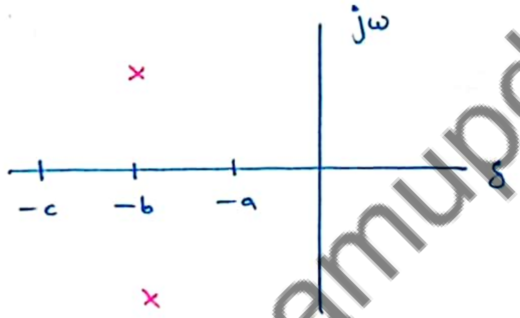
ξ	ω_1	ω_2	
0	$j\omega_n$	$-j\omega_n$	Purely imaginary
=1	$-\omega_n$	$-\omega_n$	Real & Equal
<1			Imaginary Complex Conjugate
>1			Purely real & distinct.

- Depending upon the Value of ξ, a second-order system can be set into one of the four categories:

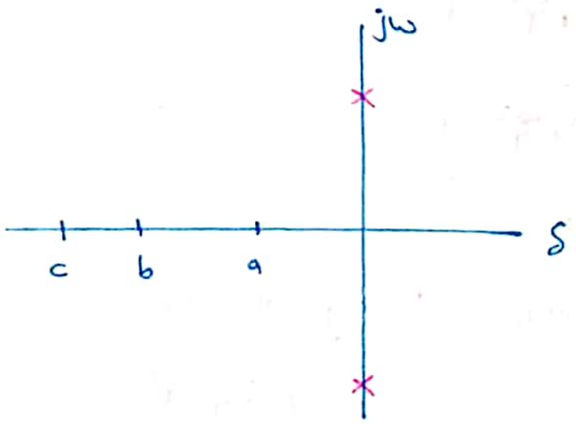
1. Over damped :- When the system has two real distinct poles $(\xi > 1)$



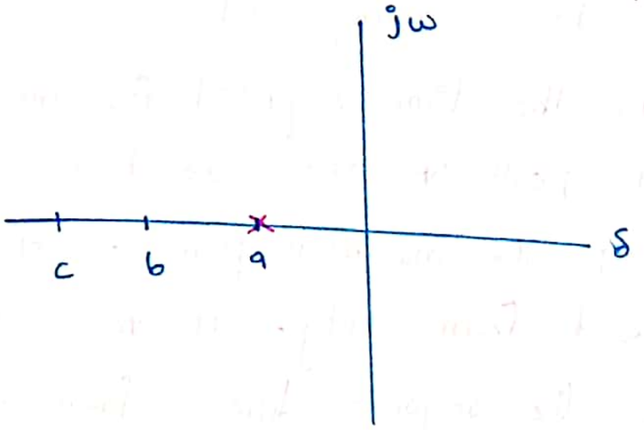
2. Under damped – When the system has two Complex Conjugate poles $(0 < \xi < 1)$



3. Undamped - When the system has two imaginary poles $(\xi = 0)$

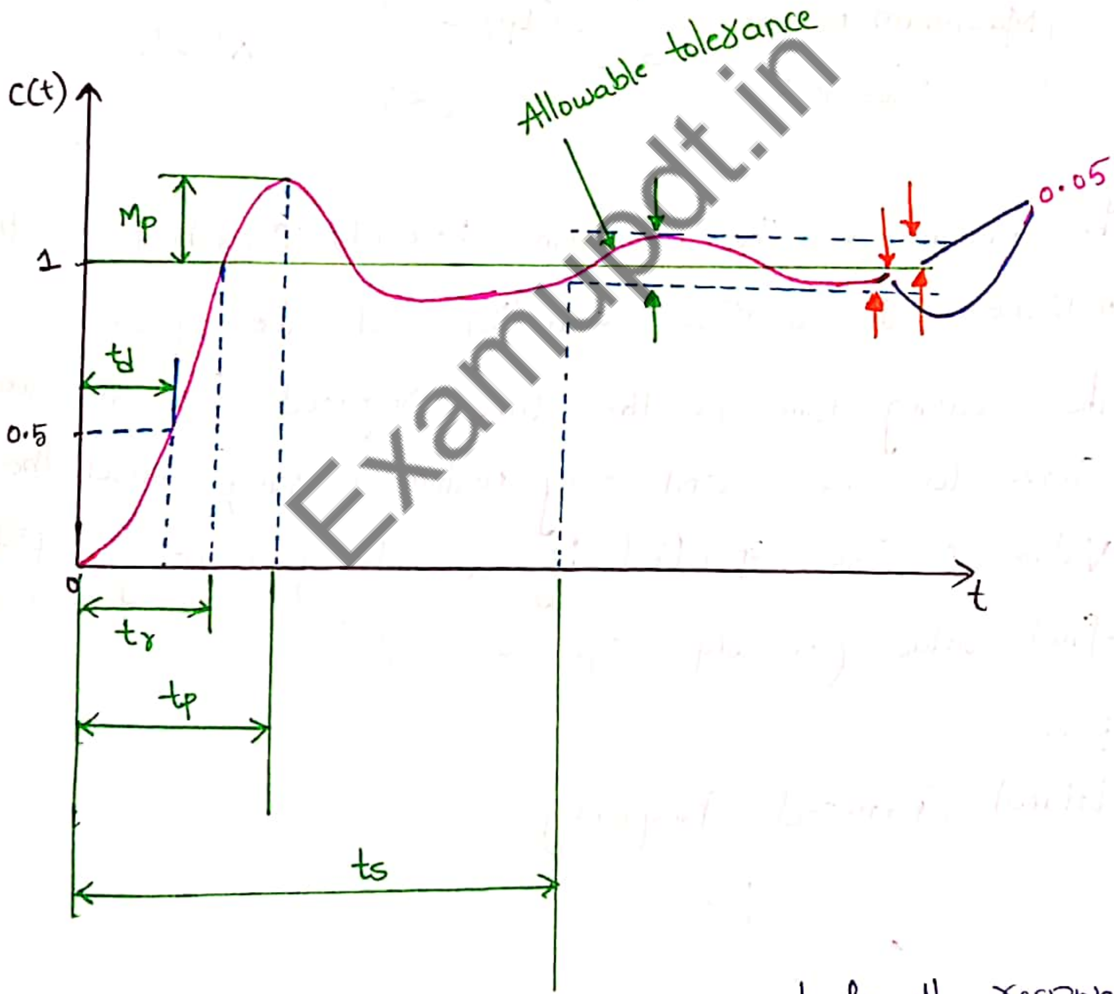


4. Critically damped – When the system has two real but equal poles ($\xi = 1$)



Time – Domain Specification

For $0<\xi<1$ and $\omega_n > 0$, the 2^{nd} order system's response due to a unit step input looks like



- The delay (t_d) time is the time required for the response to reach half the final value the very first time.
- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95% (or) 0% to 100% of its final value.

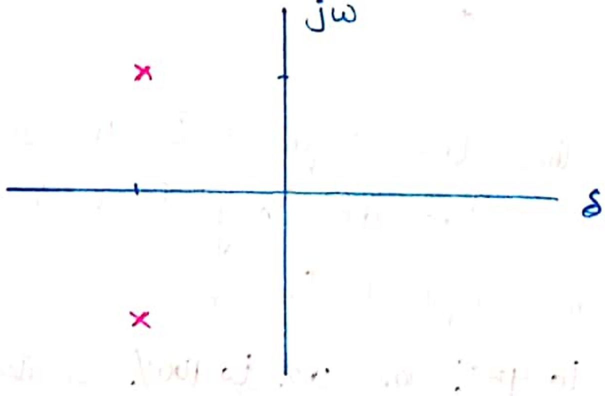
- For Underdamped Second order Systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
- The Peak time (t_p) is the time required for the response to reach the first peak of the over shoot.
- The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\boxed{\text{Maximum percent Overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%}$$

- The amount of the maximum (percent) Overshoot directly indicates the relative stability of the system.
- The Settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% (or) 5%).

S-plane.

- Natural Undamped Frequency



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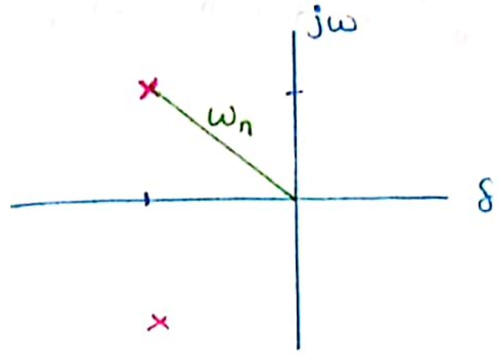
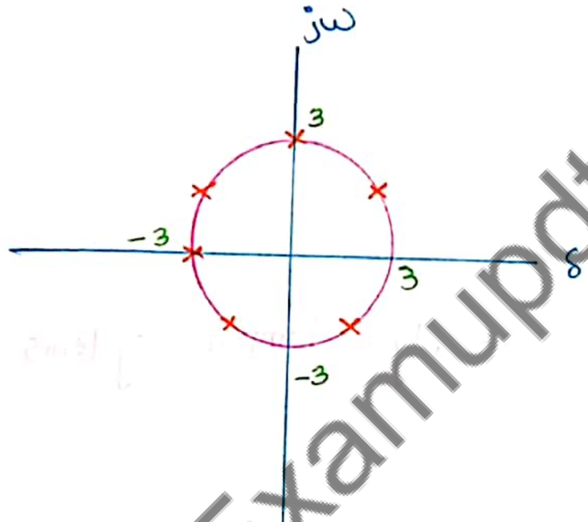
• Distance from the origin of s-plane to pole is natural Undamped frequency in rad/sec.

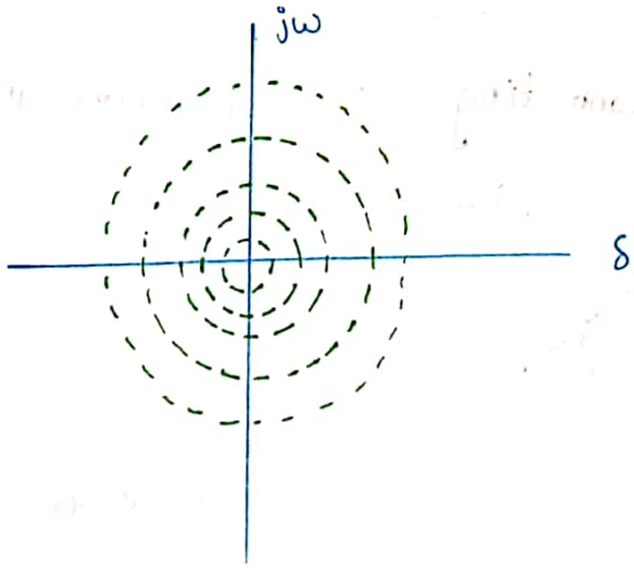


• Let us draw a circle of radius 3 in s-plane.

• If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be 3 rad/sec

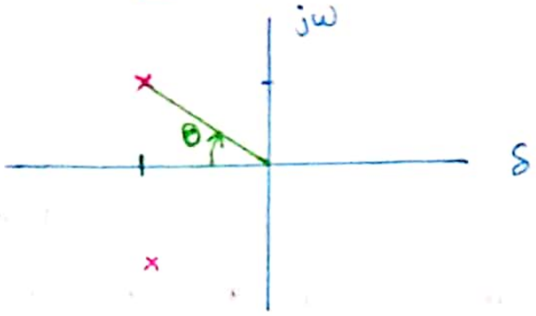


• Therefore the s-plane is divided into Constant Natural Undamped Frequency (ω_n) Circles

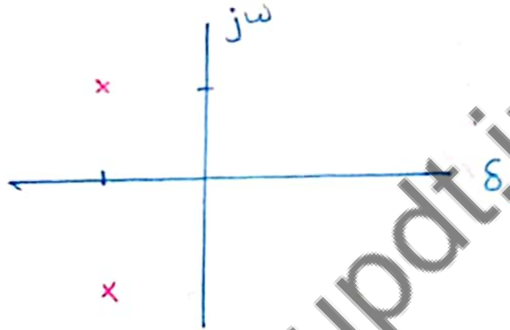


- Damping ratio.
- Cosine of the angle between Vector Connecting origin and pole and -ve real axis yields damping ratio

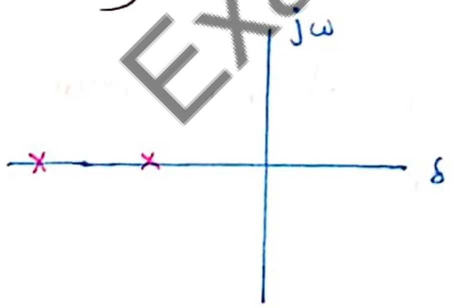
$$\boxed{\xi = \cos\theta}$$



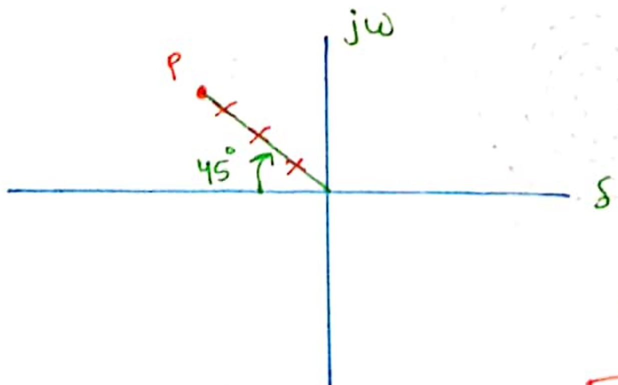
- For Underdamped System $0^\circ < \theta^\circ < 90^\circ$ therefore, $0 < \xi < 1$



- For overdamped and critically damped systems $\theta = 0^\circ$ therefore, $\xi \geq 1$



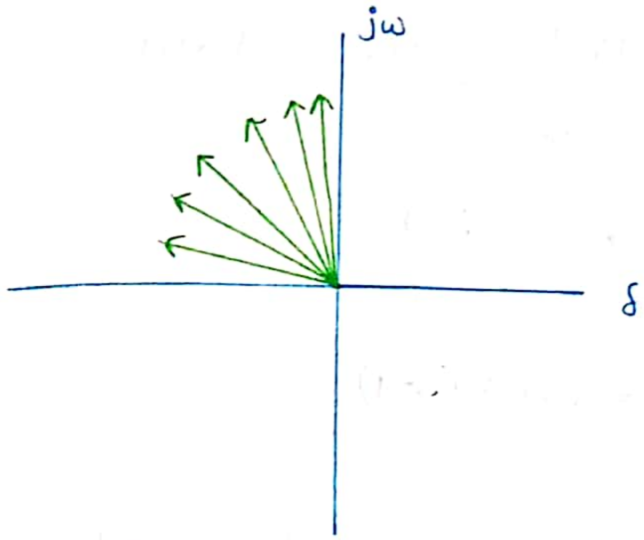
- Draw a Vector Connecting origin of S-plane and some point P.



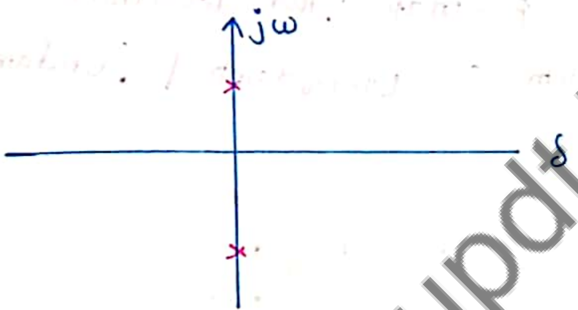
$$\xi = \cos 45^\circ$$

$$\boxed{\xi = 0.707}$$

∴ S-plane is divided into sections of constant damping ratio lines.

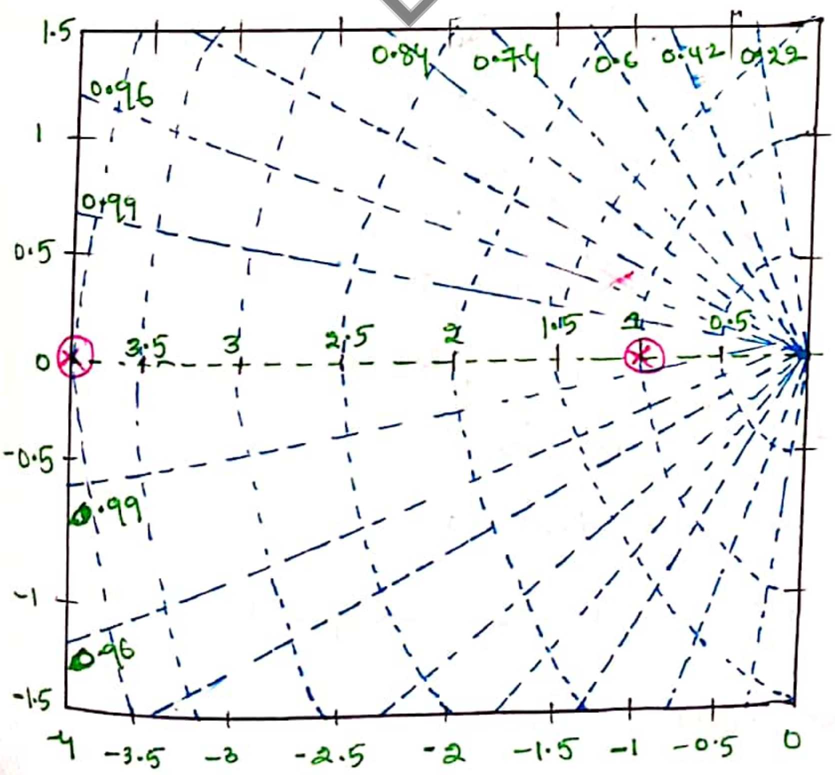


• For Undamped system $\theta = 90^\circ$ therefore $\xi = 0$



Example-2:

Determine the natural frequency and damping ratio of the poles from the P-Z map.



Two roots on negative real axis,

$\xi \geq 1$

Real and distinct $\xi > 1$, overdampet

$\omega_n = 1$ & $\omega_n = 4$

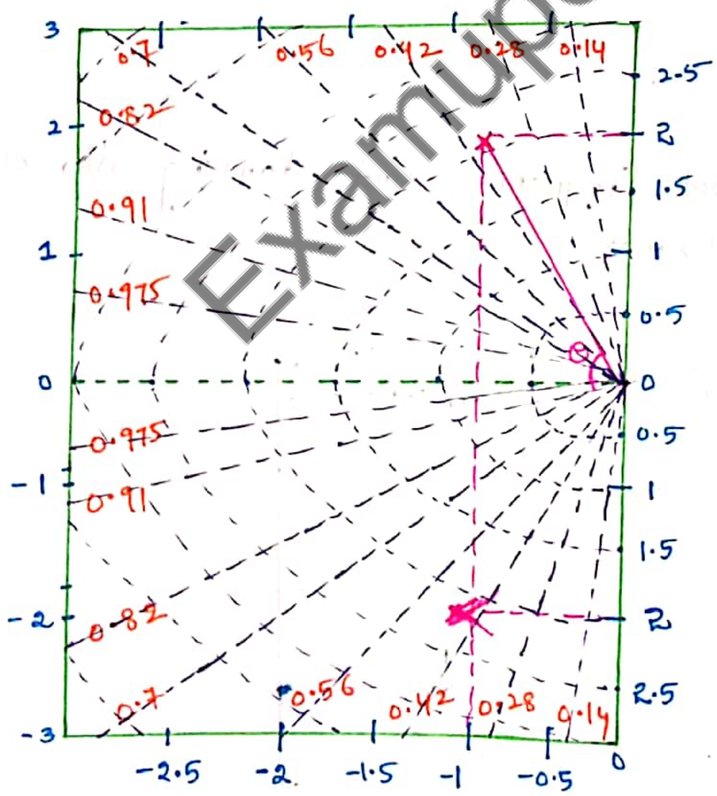
Roots $\rightarrow$ $-1+j0$, $-4+j(0)$

$s=-1$ $s=-4$

$\therefore$ $\boxed{\text{Roots} = (s+1)(s-4)}$

Example-3:

Determine the natural frequency and damping ratio of the poles from the given p-z map. Also determine the transfer function of the system is Underdamped ; Undamped (or) critically damped.



- Two roots are on -ve real axis
- Roots are complex conjugate $(0 < \xi < 1)$

$\omega_n = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

$\omega_n = 2.23$

$\xi = 0.45$ $\boxed{\cos\theta = \frac{adj}{hyp}}$

Roots $= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$= -(0.45)(2.23) \pm j(2.23)\sqrt{1-(0.45)^2}$

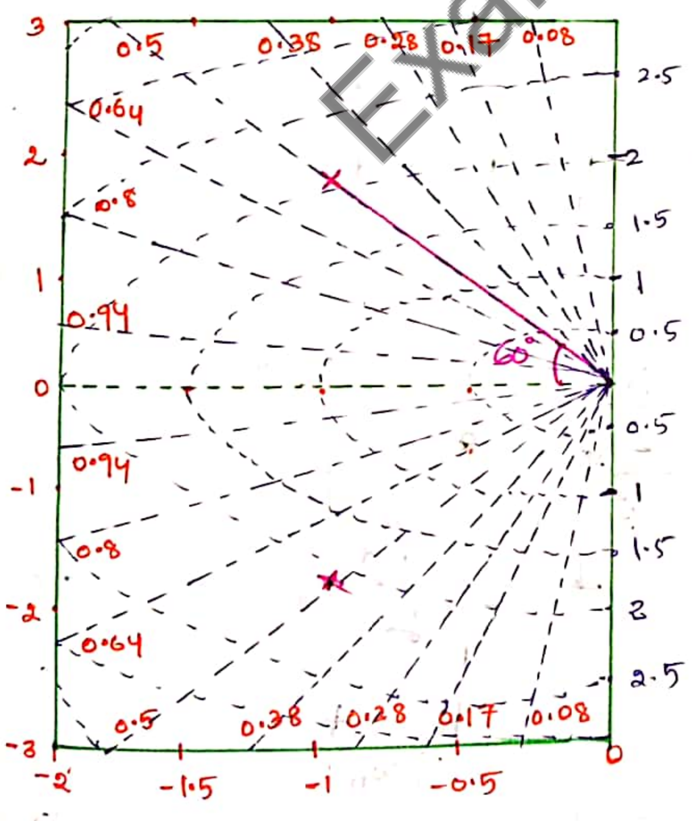
$= -1 \pm j(2.23)(0.893)$

$= -1 \pm j(1.99)$

$\therefore$ $\boxed{\text{Roots} = (-1+1.99j)\ (-1-1.99j)}$

Example-4:

Q. The natural frequency of closed loop poles of 2nd order system is 2 rad/sec and damping ratio is 0.5. Determine the location of closed loop holes so that the damping ratio remains same but the natural undamped frequency is doubled.



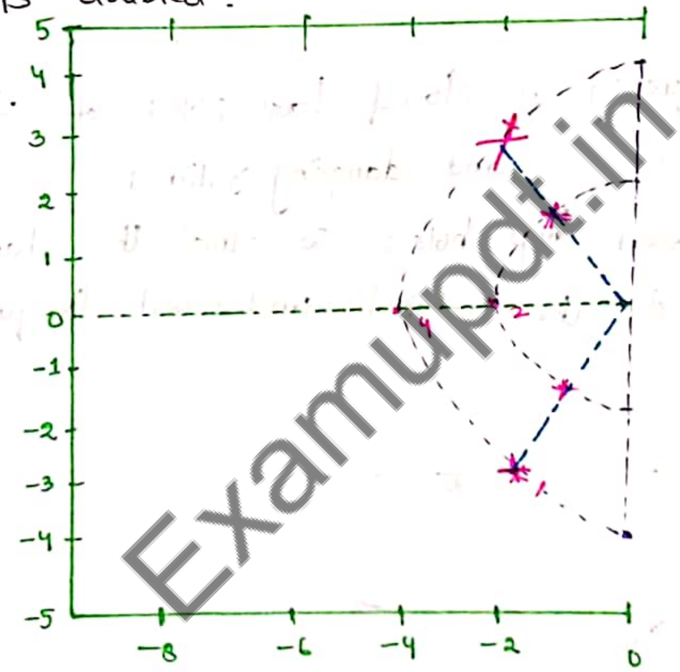
Given $\omega_n = 2, \quad \xi = 0.5$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{(2)^2}{s^2 + 2(0.5)(2)s + (2)^2}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}}$$

Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.



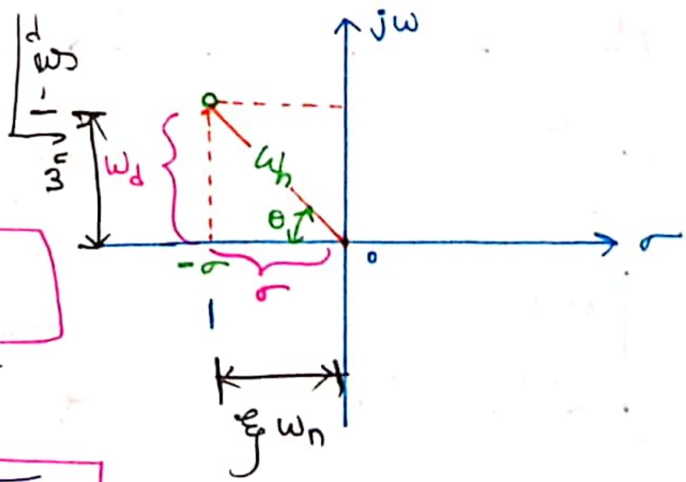
S-plane:

$$-\omega_n\xi + \omega_n\sqrt{\xi^2 - 1}$$

$$-\omega_n\xi \mp \omega_n\sqrt{\xi^2 - 1}$$

$$\sigma = \xi\omega_n$$

$$\boxed{\xi = \frac{\sigma}{\omega_n} = \cos\theta}$$



$$\omega_d = \omega_n\sqrt{1 - \xi^2}$$

$$\boxed{\frac{\omega_d}{\omega_n} = \sin\theta = \sqrt{1 - \xi^2}}$$

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Step Response of Underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \xrightarrow[\text{Response}]{\text{step}} C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

the partial fraction expression of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\xi^2\omega_n^2+\omega_n^2-\xi^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \quad \boxed{\because \omega_d = \omega_n^2(1-\xi^2)}$$

Where $\omega_d = \omega_n\sqrt{1-\xi^2}$ is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equations can be obtained easily if $C(s)$ is written in the following form.

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\frac{\xi}{(\sqrt{1-\xi^2})}\omega_n(\sqrt{1-\xi^2})}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}}\,\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

Taking inverse laplace on both sides

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}}\, e^{-\xi\omega t}\sin\omega_d t$$

$$\boxed{c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]}$$

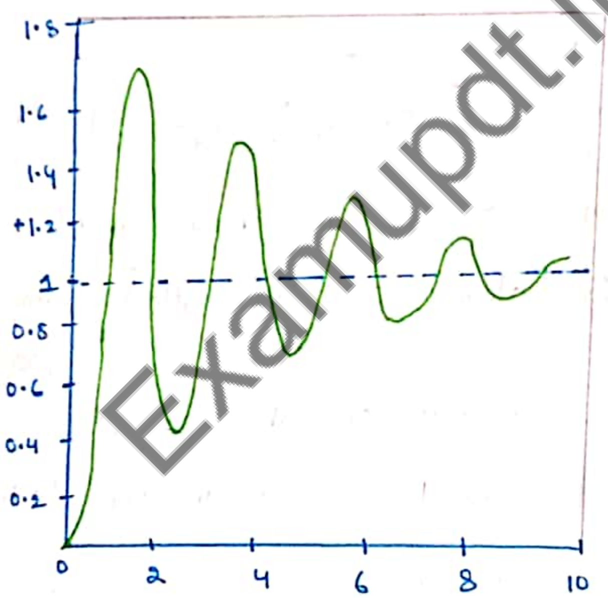
When $\xi = 0$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$$

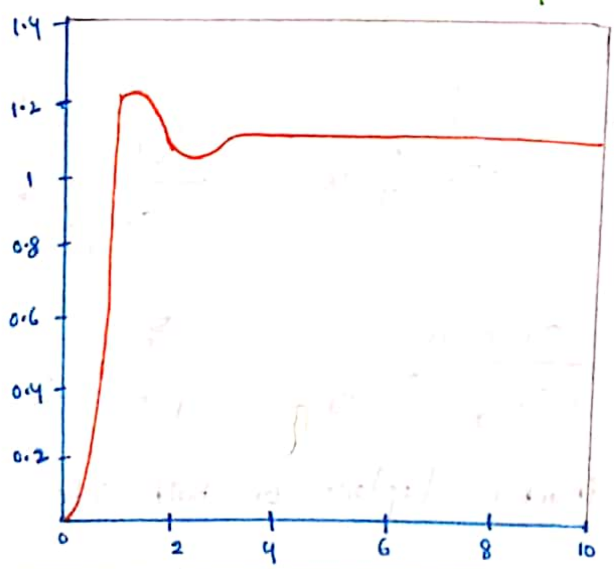
$$\boxed{c(t) = 1 - \cos\omega_n t}$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

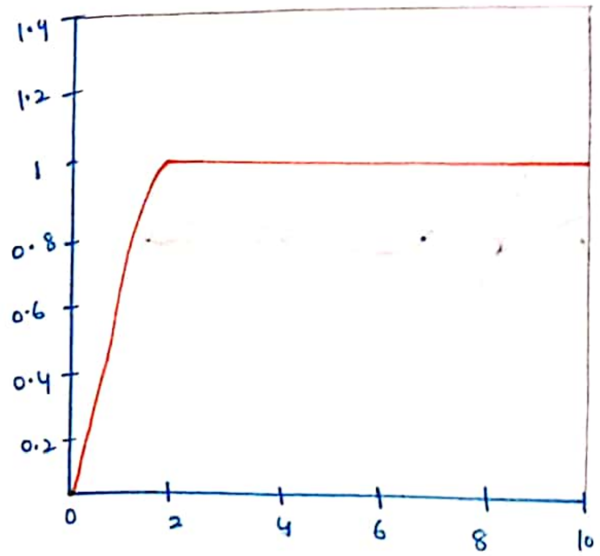
If $\xi = 0.1$ and $\omega_n = 3$ rad/sec

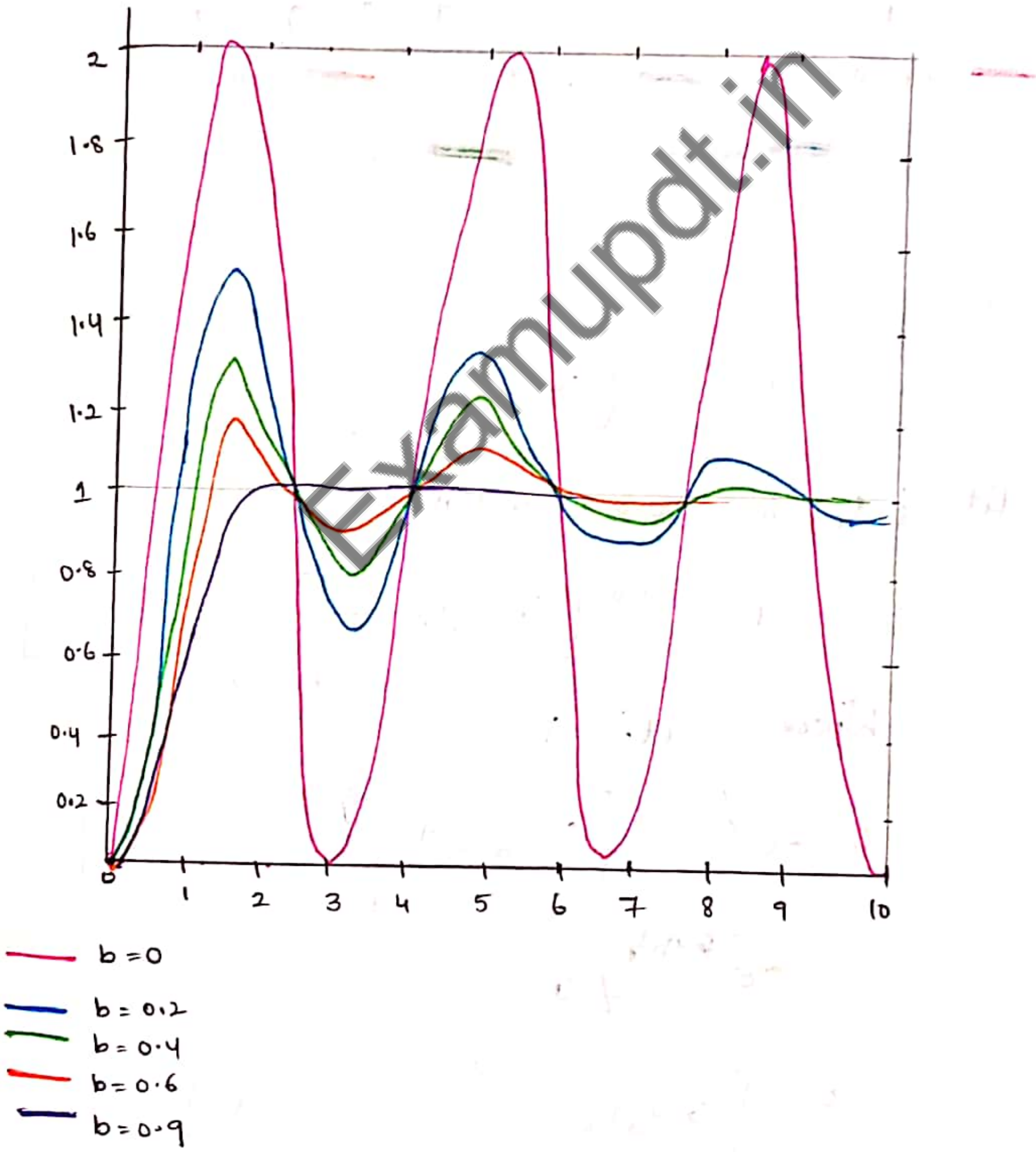


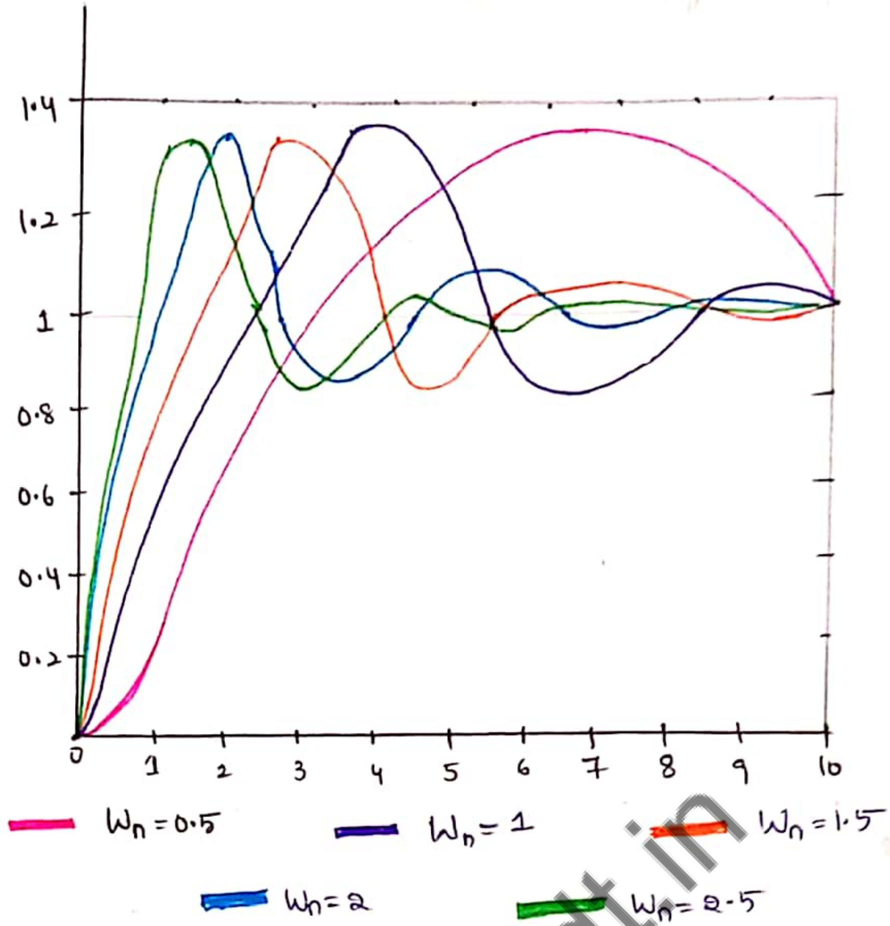
If $\xi = 0.5$ and $\omega_n = 3$ rad/sec



If $\xi = 0.9$ and $\omega_n = 3$ rad/sec







Time Domain Specifications

Rise time

$$c(t) = 1 - e^{-\xi \omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

Put $t = t_r$ in above equation

$$c(t_r) = 1 - e^{-\xi \omega_n t_r}\left[\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r\right]$$

Where $c(t_r) = 1$

$$0 = -e^{-\xi \omega_n t_r}\left[\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r\right]$$

$$-e^{-\xi \omega_n t_r} \neq 0$$

$$0 = \left[\cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_r\right]$$

Above equation can be written as

$$\sin\omega_d t_r = -\frac{\sqrt{1-\xi^2}}{\xi}\cos\omega_d t_r$$

$$\tan\omega_d t_r = -\frac{\sqrt{1-\xi^2}}{\xi}$$

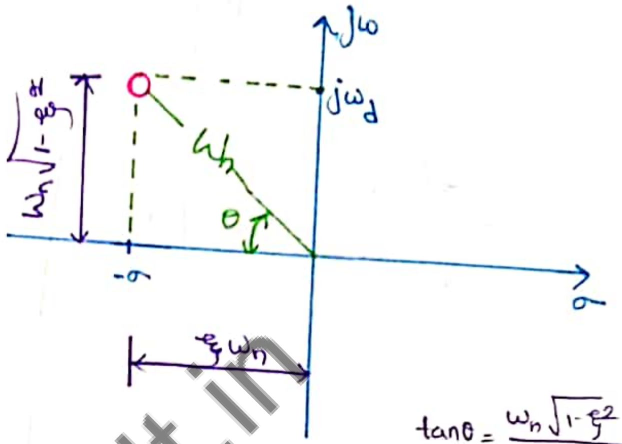
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_n\sqrt{1-\xi^2}}{\omega_n\xi}\right)$$

$$\therefore \boxed{t_r = \frac{\pi-\theta}{\omega_d}}$$

$$\therefore \boxed{t_r = \frac{\pi-\cos^{-1}(\xi)}{\omega_n\sqrt{1-\xi^2}}}$$



$$\tan\theta = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{(\xi)}\right)$$

Peak time:

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

In order to find peak time let us differentiate above equation with respect to time.

$$\frac{dc(t)}{dt} = \xi\omega_n e^{-\xi\omega_n t}\left[\cos\omega_d + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right] - e^{-\xi\omega_n t}\left[-\omega_d\sin\omega_d t + \frac{\xi\omega_d}{\sqrt{1-\xi^2}}\cos\omega_d t\right]$$

$$0 = e^{-\xi\omega_n t}\left[\xi\omega_n\cos\omega_d t + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\xi\omega_d}{\sqrt{1-\xi^2}}\cos\omega_d t\right]$$

$$0 = e^{-\xi\omega_n t}\left[\xi\omega_n\cos\omega_d t + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\xi\omega_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}\cos\omega_d t\right]$$

$$0 = e^{-\xi\omega_n t}\left[\xi\omega_n\cos\omega_d t + \frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \xi\omega_n\cos\omega_d t\right]$$

$$e^{-\xi\omega_n t}\left[\frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t+\omega_d\sin\omega_d t\right]=0$$

$$e^{-\xi\omega_n t}\neq 0$$

$$\left[\frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t+\omega_d\sin\omega_d t\right]=0$$

$$\sin\omega_d t\left[\frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}+\omega_d\right]=0$$

$$\frac{\xi^2\omega_n}{\sqrt{1-\xi^2}}+\omega_d\neq 0,$$

$$\sin\omega_d t=0$$

$$\omega_d t=\sin^{-1}(0)$$

$$\boxed{t=\frac{0,\pi,2\pi,\cdots}{\omega_d}}$$

Since for Underdamped stable systems first peak is maximum peak. Therefore,

$$\therefore \boxed{t_p=\frac{\pi}{\omega_d}}$$

Maximum Overshoot.

$$\text{Maximum percent Overshoot}=\frac{c(t_p)-c(\infty)}{c(\infty)}\times 100\%$$

$$c(t_p)=1-e^{-\xi\omega_n t_p}\left[\cos\omega_d t_p+\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p\right]$$

$$c(\infty)=1$$

$$M_p=\left[1-e^{-\xi\omega_n t_p}\left[\cos\omega_d t_p+\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p\right]-1\right]\times 100$$

put $t_p=\frac{\pi}{\omega_d}$ in above equation.

$$M_p = \left[-e^{-\xi\omega_n \frac{\pi}{\omega_d}} \left[\cos\omega_d \frac{\pi}{\omega_d} + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_d \frac{\pi}{\omega_d} \right] \right] \times 100$$

$$M_p = \left[-e^{-\xi\omega_n \frac{\pi}{\omega_d}} \left[\cos\pi + \frac{\xi}{\sqrt{1-\xi^2}} \sin\pi \right] \right] \times 100$$

Put $\omega_d = \omega_n\sqrt{1-\xi^2}$ in above equation

$$M_p = \left[-e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \left[-1 + \frac{\xi}{\sqrt{1-\xi^2}} (0) \right] \right] \times 100$$

$$M_p = \left[-e^{-\xi \frac{\pi}{\sqrt{1-\xi^2}}} (-1) \right] \times 100$$

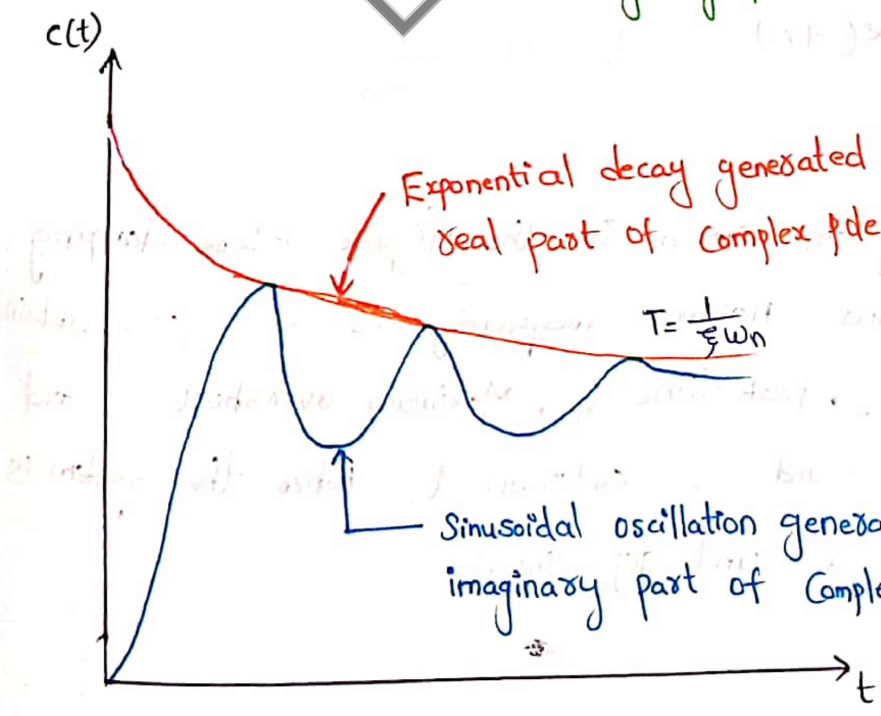
$$\therefore \boxed{M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100}$$

Settling Time

$$c(t) = 1 - e^{-\xi\omega_n t} \left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_d t \right]$$

$$-\omega_n\xi \pm \omega_n\sqrt{\xi^2 - 1}$$

Real part   Imaginary part



* Settling time (2%) criterion

Time consumed in exponential decay up to 98% of the input

$$\boxed{t_s = 4T = \frac{4}{\xi \omega_n}}$$

* Settling time (5%) criterion

Time consumed in exponential decay up to 95% of the input

$$\boxed{t_s = 3T = \frac{3}{\xi \omega_n}}$$

Summary of Time Domain Specifications.

Rise time $\quad t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\xi^2}}$

Peak time $\quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$

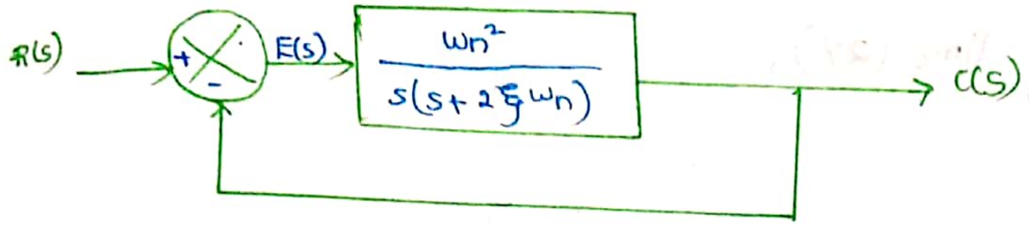
Maximum Overshoot $\quad M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$

Settling Time (2%) $\quad t_s = 4T = \frac{4}{\xi \omega_n}$

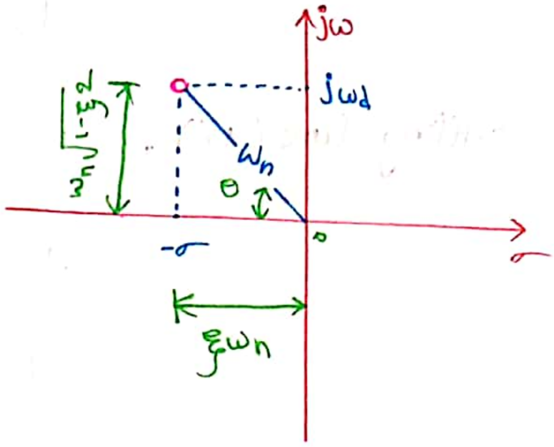
Settling Time (4%) $\quad t_s = 3T = \frac{3}{\xi \omega_n}$

Example-5 :

Consider the system shown in the figure, where damping ratio is 0.6 and natural frequency is 5 rad/sec. obtain the rise time t_r, Peak time t_p, Maximum overshoot M_p and Settling time 2% and 5% criterion t_s when the system is subjected to a unit-step input.



Ans: Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d}$



$$t_r = \frac{3.141 - \theta}{\omega_n\sqrt{1-\xi^2}}$$

$\xi = 0.6$, $\omega_n = 5$ rad/sec

$$\theta = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n}\right)$$

$$\theta = 0.93 \text{ rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1-(0.6)^2}}$$

$$\boxed{t_r = 0.55\ S}$$

Maximum over shoot, $M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$

$$= e^{\frac{-3.14(0.6)}{\sqrt{1-(0.6)^2}}} \times 100$$

$$= 0.095 \times 100$$

$$\boxed{M_p = 9.5\%}$$

Peak time, $t_p = \dfrac{\pi}{\omega_d}$

$$t_p = \frac{3.141}{4}$$

$$\boxed{t_p = 0.785\ S}$$

Settling time (2%), $t_s = \frac{4}{\xi \omega_n}$

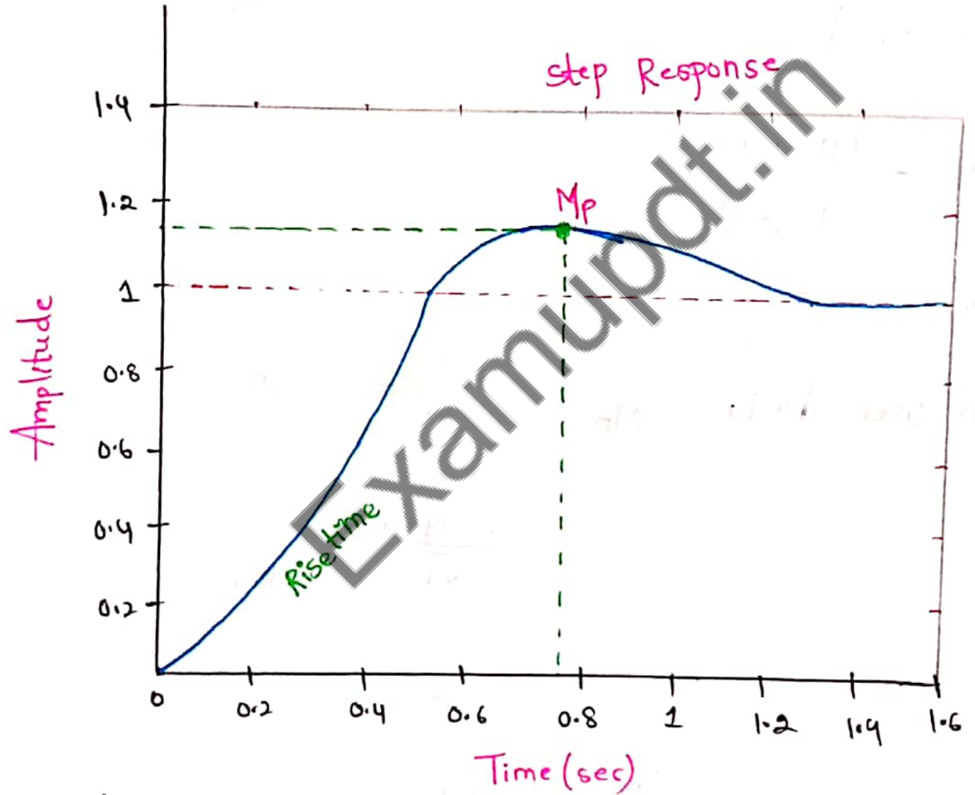
$$t_s = \frac{4}{0.6 \times 5}$$

$$\boxed{t_s = 1.33\,s}$$

Settling time (4%), $t_s = \frac{3}{\xi \omega_n}$

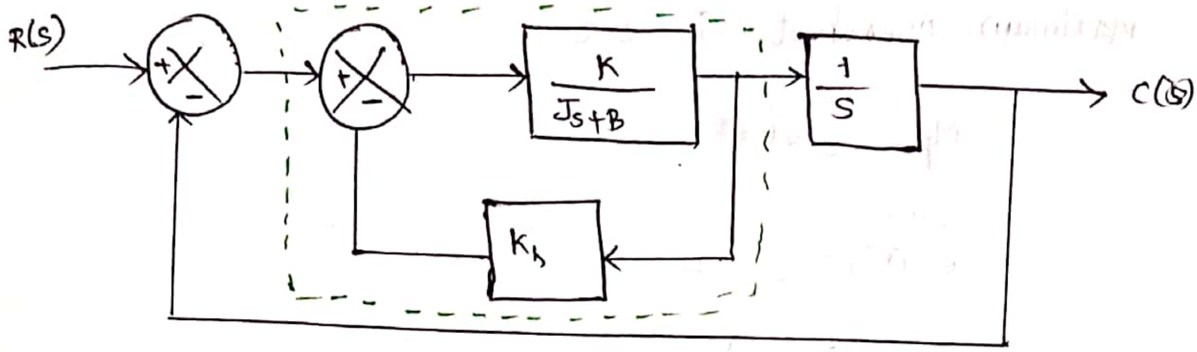
$$t_s = \frac{3}{0.6 \times 5}$$

$$\boxed{t_s = 1\ sec}$$



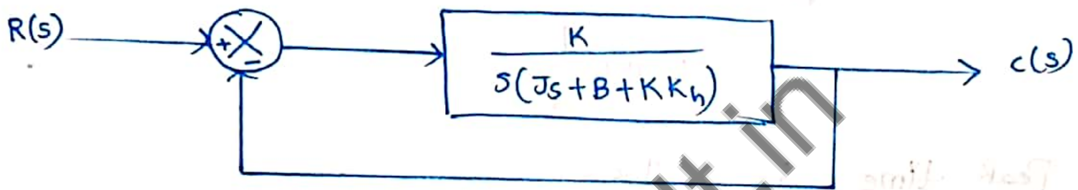
Example-6:

For the system shown in figure, determine the Values of gain K and velocity-feedback Constant K_h So that the maximum Overshoot in the unit-step response is 0.2 and the peak time is 1 sec. with these Values of K and K_h Obtain the rise time and Settling time. Assume that $J = 1\,kg\,m^2$ and $B = 1\,N\text{-}m/rad/sec$



$$\frac{G}{1+GH} = \frac{\frac{K}{Js+B}}{1+\frac{KK_h}{Js+B}} = \frac{K}{s(Js+B+KK_h)}$$

$$G_1G_2 = \frac{K}{Js+B+KK_h}\left(\frac{1}{s}\right)$$



$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+(B+KK_h)s+K}$$

Since $J = 1\,kgm^2$ and $B = 1\,Nm/rad/sec$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+(1+KK_h)s+k}$$

Comparing above T.F with general 2^{nd} order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\omega_n^2 = K$

$$\boxed{\omega_n = \sqrt{K}}$$

$$2\xi\omega_n s = (1+KK_h)s$$

$$\xi = \frac{1+KK_h}{2\omega_n}$$

$$\boxed{\xi = \frac{1+KK_h}{2\sqrt{K}}}$$

Maximum overshoot is 0.2

$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

$$e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 0.2$$

$$\ln\left(e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\right) = \ln(0.2)$$

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = -1.61$$

$$\frac{\pi\xi}{\sqrt{1-\xi^2}} = 1.61$$

$$\boxed{\xi = 0.456}$$

Peak time is 1sec

$$t_p = \frac{\pi}{\omega_d}$$

$$\therefore 1 = \frac{3.14}{\omega_n\sqrt{1-\xi^2}}$$

$$\omega_n = \frac{3.414}{\sqrt{1-(0.456)^2}}$$

$$\boxed{\omega_n = 3.53}$$

$$\omega_n = \sqrt{K}$$

$$3.53 = \sqrt{K}$$

$$K = (3.53)^2$$

$$K = 1.25$$

$$\xi = \frac{1+KK_h}{2\sqrt{K}}$$

$$0.456 = \frac{1+12.5K_h}{2(3.53)}$$

$$0.456(2)(3.53) = 1 + 1.25\,K_h$$

$$3.21936 - 1 = 1.25\,K_h$$

$$\boxed{K_h = 1.775}$$

Rise time,

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}$$

$$\boxed{t_r = 0.65\ \text{sec}}$$

settling time (2%),

$$t_s = \frac{4}{\xi\omega_n}$$

$$t_s = \frac{4}{(0.456)(3.53)}$$

$$\boxed{t_s = 2.48\ \text{sec}}$$

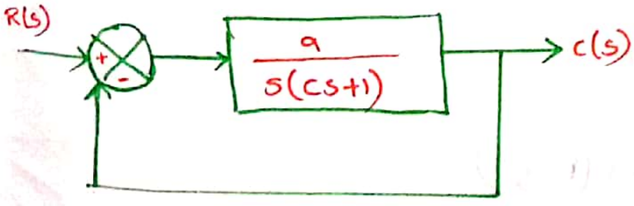
settling time (4%),

$$t_s = \frac{3}{\xi\omega_n}$$

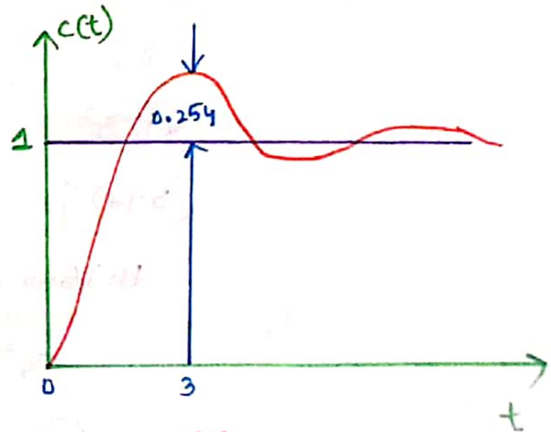
$$\boxed{t_s = 1.86\ \text{sec}}$$

Example- 7:

When the system shown in figure (a) is subjected to a unit-step input, the system output responds as shown in fig (b). Determine the values of a and c from the response curve.



(a)



(b)

$$\frac{C(s)}{G(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{a}{s(cs+1)}}{1+\frac{a}{s(cs+1)}}$$

$$= \frac{a}{cs^2+s+a}$$

$$= \frac{\frac{a}{c}}{s^2+\frac{1}{c}s+\frac{a}{c}}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{\frac{a}{c}}{s^2+\frac{1}{c}s+\frac{a}{c}}}$$

Maximum overshoot $M_p = 0.254$ (Given from graph)

Peak time $t_p = 3$ sec (Graph).

$$M_p = 0.254$$

$$e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 0.254$$

Taking log on both sides

$$\ln\left(e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\right) = \ln(0.254)$$

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = -1.37$$

$$\frac{\pi\xi}{\sqrt{1-\xi^2}} = 1.37$$

$$(3.14)\xi = 1.37(1-\xi^2)$$

$$11.7465\,\xi^2 = 1.8769$$

$$\xi^2 = 0.097$$

$$\boxed{\xi = 0.3997} \quad \text{Eq } 4$$

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\xi^2}}$$

$$= \frac{3.14}{3\sqrt{1-(0.4)^2}}$$

$$\boxed{\omega_n = 1.1425 \text{ rad/sec}}$$

$$\omega_n^2 = \frac{a}{c} \quad ; \quad 2\xi\omega_n s = \frac{1}{c}s$$

$$c = \frac{1}{2\xi\omega_n}$$

$$= \frac{1}{2\times 0.4\times 1.1425}$$

$$\boxed{c = 1.094}$$

$$a = c\omega_n^2$$

$$= 1.094(1.1425)^2$$

$$\boxed{a = 1.428}$$

Step Response of critically damped System ($\xi = 1$)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2} \xrightarrow[\text{Response}]{\text{step}} C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

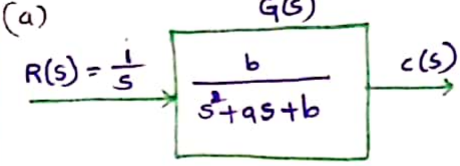
Taking inverse laplace on both sides

$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t}\, t$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)}$$

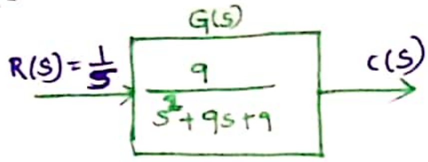
System	Pole-Zero plot	Response

(a)



General

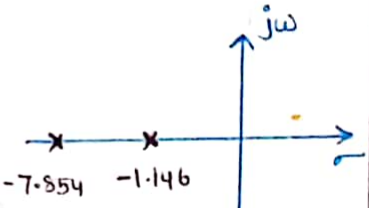
(b)



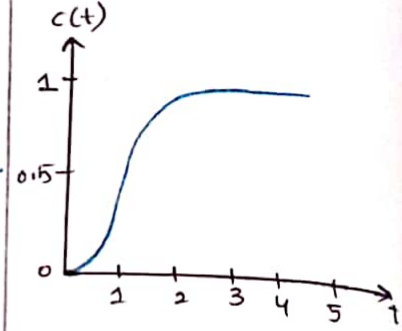
Overdamped

$\omega_n^2 = 9 \Rightarrow \omega_n = 3$

$2\xi\omega_n s = 9s \Rightarrow \xi = 1.5$



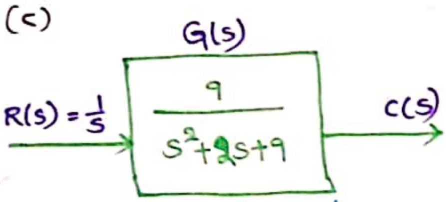
s-plane



$c(t) = 1 + 0.17\left[e^{-7.854t}\right]t$

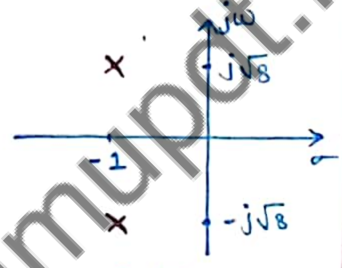
$- 1.17\,t\,e^{1.146t}$

(c)

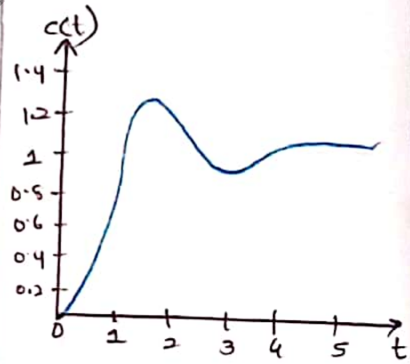


Under damped

$\omega_n = 3$; $\xi = 0.333$



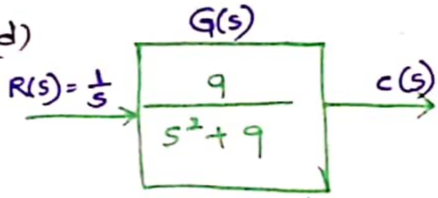
s-plane



$c(t) = 1 - e^{-t}\left(\cos\sqrt{8}\,t + \frac{\sqrt{8}}{8}\sin\sqrt{8}\,t\right)$

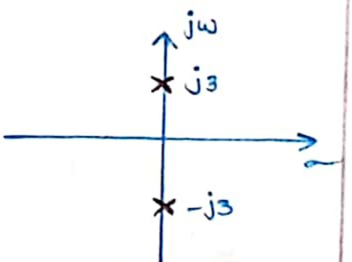
$= 1 - 1.06\,e^{-t}\cos(\sqrt{8}\,t - 19.47^\circ)$

(d)

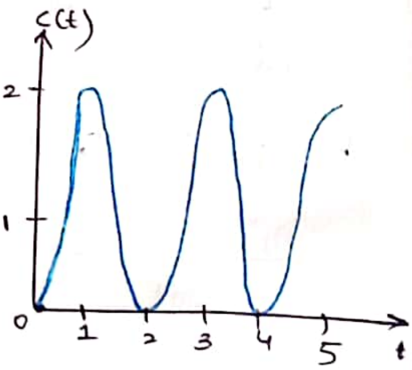


Undamped

$\omega_n = 3$; $\xi = 0$

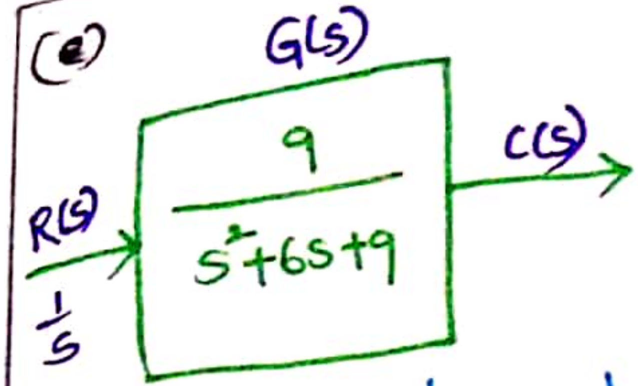


S-plane



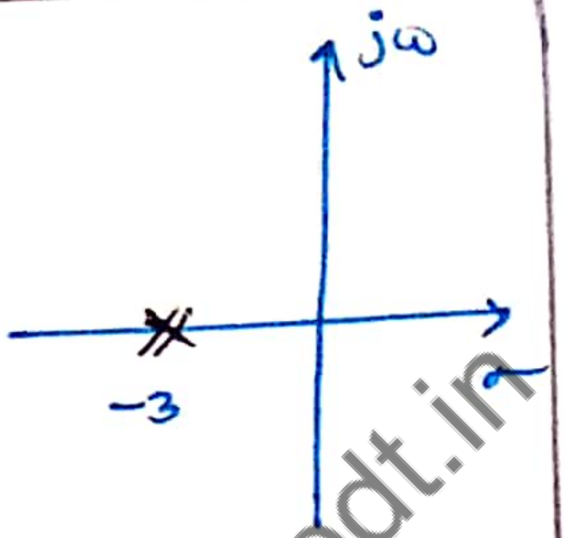
$c(t) = 1 - \cos 3t$

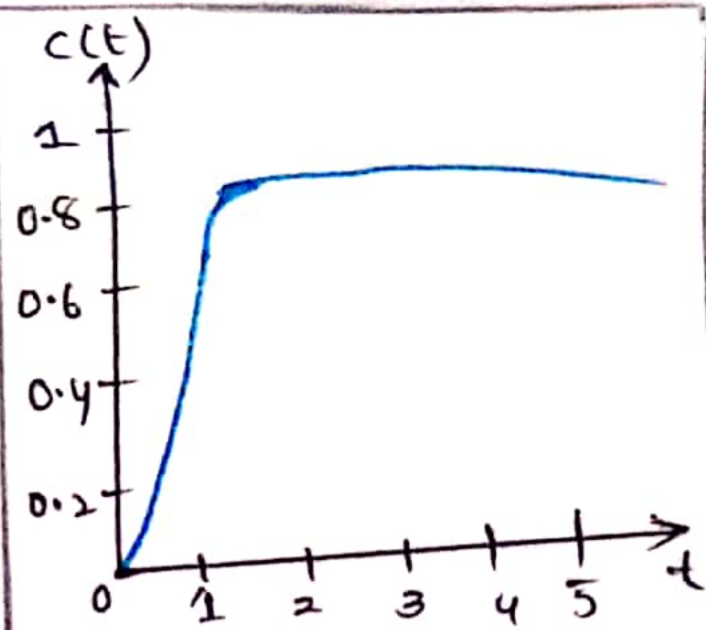
(e)



critically damped

$\xi = 1$

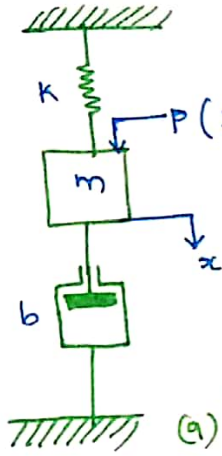




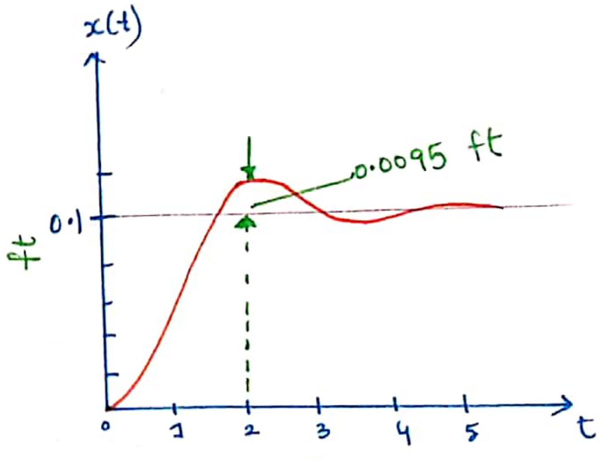
$$c(t) = 1 - 3t\,e^{-3t} - e^{-3t}$$

Homework:

Figure (a) shows a mechanical Vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in figure (b). Determine m, b, and K of the system from this response.



(a)



(b)

Sol:- Transfer function of the System,

$$\frac{x(s)}{P(s)} = \frac{1}{(Ms^2+bs+k)}$$

$$\because \ P(s) = \frac{2}{s}$$

$$x(s) = \frac{2}{s(Ms^2+bs+K)}$$

Steady state Value of x is

$$x(\infty) = \lim_{s\to 0} s\,x(s) = \frac{2}{K} = 0.1\,ft$$

$$\therefore \boxed{K = 20\ lb_f/ft}$$

Note that $M_p = 9.5\%$ Corresponds to $\xi = 0.6$.

Peak time t_p is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{0.8\,\omega_n}$$

$$t_p = 2\,sec,$$

$$\omega_n = \frac{\pi}{0.8\, t_p}$$

$$= \frac{\pi}{0.8(2)}$$

$$\boxed{\omega_n = 1.96 \text{ rad/sec}}$$

$$\omega_n^2 = \frac{K}{m} = \frac{20}{m}$$

$$m = \frac{20}{\omega_n^2}$$

$$= \frac{20}{(1.96)^2}$$

$$\boxed{m = 5.2\ lb} \qquad \boxed{m = 5.2\ lb}$$

$$\boxed{1 \text{ slug} = 1\ lb_t\ sec/ft}$$

Then b is

$$2\xi\omega_n = \frac{b}{m}$$

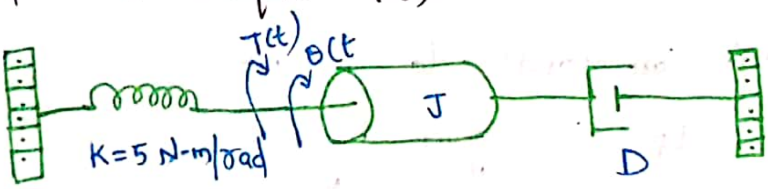
$$b = 2\xi\omega_n m$$

$$= 2(1.96)(0.6)(5.2)$$

$$\boxed{b = 12.2\ lb_t/ft/sec}$$

Example-9:

Given the system shown in following figure, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque T(t).



Sol:- First, the transfer function for the system is

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$2\xi\omega_n = \frac{D}{J}$$

Given,

$$T_s = 2 = \frac{4}{\xi\omega_n}$$

$$\boxed{\xi\omega_n = 2}$$

$$2\xi\omega_n = \frac{D}{J}$$

$$\frac{D}{J} = 4$$

$$\xi = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}}$$

20% Overshoot impulse $\xi = 0.456$

$$\xi = 2\sqrt{\frac{J}{K}} = 0.456$$

$$\frac{J}{K} = 0.052$$

Given, $K = 5$ N-m/rad

$$J = 0.052 \times K$$

$$J = 0.052 \times 5$$

$$\boxed{J = 0.26 \text{ Kg-m}^2}$$

$$D = 4J$$

$$D = 4(0.26)$$

$$\boxed{D = 1.04 \text{ N-m/rad}}$$

Example-10

Describe the nature of the second-order system response via the value of the damping ratio for the systems with transfer function.

1) $G(s) = \dfrac{12}{s^2+8s+12}$

Sol: Compare with $\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 12 \Rightarrow \boxed{\omega_n = 3.464}$

$2\xi\omega_n s = 8s \Rightarrow \boxed{\xi = 1.15}$

Overdamped

2) $G(s) = \dfrac{16}{s^2+8s+16}$

Sol: Compare with $\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 16$

$\omega_n = 4$

$2\xi\omega_n s = 8s$

$2\xi(4)s = 8s$

$\xi = 1$

Critically damped

3) $G(s) = \dfrac{20}{s^2+8s+20}$

Sol: Compare with $\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 20$

$\omega_n = 4.472$

$2\xi\omega_n s = 8s$

$\xi = 0.894$

Under damped

Example-10

Describe the nature of the Second-order system response via the value of the damping ratio for the systems with transfer function.

1) $G(s) = \frac{12}{s^2+8s+12}$

Sol: Compare with $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 12 \Rightarrow \boxed{\omega_n = 3.464}$

$2\xi\omega_n s = 8s \Rightarrow \boxed{\xi = 1.15}$

Overdamped

2) $G(s) = \frac{16}{s^2+8s+16}$

Sol: Compare with $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 16$ $\quad$ $2\xi\omega_n s = 8s$

$\omega_n = 4$ $\quad$ $2\xi(4)s = 8s$

$\xi = 1$

Critically damped

3) $G(s) = \frac{20}{s^2+8s+20}$

Sol: Compare with $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 20$ $\quad$ $2\xi\omega_n s = 8s$

$\omega_n = 4.472$ $\quad$ $\xi = 0.894$

Under damped

Steady-state error

Introduction:

- Any physical Control system inherently suffers steady-state error in response to certain types of inputs.
- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.
- Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system

Classification of Control systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, ~~according~~ parabolic inputs and so on.
- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system
- Consider the unity-feedback control system with the following open-loop transfer function.

$$G(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

* It involves the term s^N in the denominator, representing N poles at the origin

* A system is called type 0, type 1, type 2, ..., if $N=0, N=1, N=2, \ldots$, respectively.

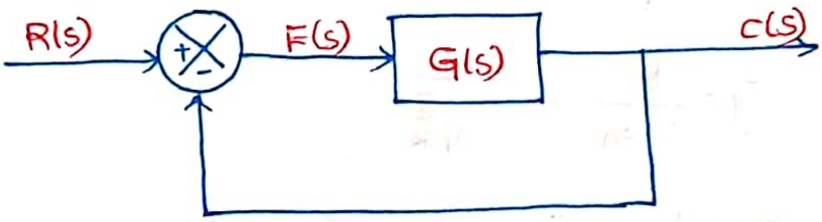
Order – Highest power of s in characteristic equation

Type – Lowest power of s. (No. of poles)

- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary

Steady state Error of Unity Feedback Systems

- Consider the system shown in following figure



The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(s) = R(s) - C(s)$$

$$= R(s) - \frac{G(s)}{1+G(s)} R(s)$$

$$= \left[\frac{1+G(s) - G(s)}{1+G(s)}\right] R(s)$$

$$\therefore \boxed{\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}}$$

- The transfer function between the error signal E(s) and the input signal R(s) is

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

The final Value theorem provides a Convenient way to find the steady-state performance of a stable system.

Since $E(s)$ is

$$\boxed{E(s) = \frac{1}{1+G(s)} R(s)}$$

The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}}$$

Static Error Constants

- The static error Constants are figures of merit of control systems. The higher the Constants, the smaller the steady-state error.
- In a given system, the output may be the position, Velocity, Pressure, temperature (or) the like.
- Therefore, in what follows, we shall call the output "position", the rate of change of the output "Velocity", and so on.
- This means that in a temperature control system "position" represents the output temperature, "Velocity" represents the rate of change of the output temperature and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s}$$

$$\boxed{e_{ss} = \frac{1}{1+G(0)}}$$

The static position error Constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s)$$

$$\boxed{K_p = G(0)}$$

Thus, the steady-state errors in terms of the static position error Constant K_p is given by

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

For a Type 0 system,

$$\boxed{K_p = \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = K}$$

For a Type 1 (or) higher systems

$$\boxed{K_p = \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty}$$

for $N \geq 1$

For a unit step input the steady state error e_{ss} is

$e_{ss} = \frac{1}{1+K}$	for type 0 Systems
$e_{ss} = 0$	for type 1 (or) higher systems

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \rightarrow \text{Polynomial representation}$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+P_1)(s+P_2)\cdots(s+P_n)} \rightarrow \text{Pole-zero Form}$$

$$G(s) = \frac{K(T_{z_1} s+1)(T_{z_2} s+1)\cdots(T_{z_m} s+1)}{(T_{P_1} s+1)(T_{P_2} s+1)\cdots(T_{P_m} s+1)} \rightarrow \text{Time Constant Form}$$

Static Velocity Error Constant (K_v)

The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^2}$$

$$= \lim_{s\to 0} \frac{1}{s[1+G(s)]}$$

$$= \lim_{s\to 0} \frac{1}{s + sG(s)}$$

$$= \frac{1}{0 + \lim_{s\to 0} sG(s)}$$

$$\boxed{e_{ss} = \lim_{s\to 0} \frac{1}{sG(s)}}$$

The static position error constant K_v is defined by

$$\boxed{K_v = \lim_{s\to 0} sG(s)}$$

Thus, the steady-state errors in terms of the static velocity error constant K_v is given by

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

For a Type 0 system,

$$\boxed{K_v = \lim_{s\to 0} \frac{sK(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0}$$

For a Type 1 system,

$$\boxed{K_v = \lim_{s\to 0} \frac{sK(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = K}$$

For type 2 (or) higher systems,

$$\boxed{K_v = \lim_{s\to 0} \frac{sK(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty} \quad \text{for } N \geq 2$$

For a ramp input the steady state error e_{ss} is

$$\boxed{e_{ss} = \frac{1}{K_v} = \infty}$$ for type 0 systems

For type 1 systems

$$\boxed{e_{ss} = \frac{1}{K_v} = \frac{1}{K}}$$

For type 2 (or) higher systems

$$\boxed{e_{ss} = \frac{1}{K_v} = 0}$$

Static Acceleration Error Constant (K_a)

The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \frac{1}{s^3}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1+G(s)} \frac{1}{s^2}$$

$$\boxed{e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s)}}$$

The static acceleration error constant K_a is defined by

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s)}$$

Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

For a Type 0 system,

$$\boxed{K_a = \lim_{s \to 0} \frac{s^2 K (T_a s+1)(T_b s+1) \cdots}{(T_1 s+1)(T_2 s+1) \cdots} = 0}$$

For Type 1 Systems,

$$\boxed{K_a = \lim_{s\to 0} \frac{s^2K(T_as+1)(T_bs+1)\cdots}{s(T_1s+1)(T_2s+1)\cdots} = 0}$$

For Type 2 Systems,

$$\boxed{K_a = \lim_{s\to 0} \frac{s^2K(T_as+1)(T_bs+1)\cdots}{s^2(T_1s+1)(T_2s+1)\cdots} = K}$$

For Type 3 (or) higher systems,

$$\boxed{K_a = \lim_{s\to 0} \frac{s^2K(T_as+1)(T_bs+1)\cdots}{s^N(T_1s+1)(T_2s+1)\cdots} = \infty} \quad \text{for } N\geq 3$$

For a parabolic input the steady-state error e_{ss} is

For type 0 and type 1 systems

$$\boxed{e_{ss} = \infty}$$

For type 2 systems

$$\boxed{e_{ss} = \frac{1}{K}}$$

For type 3 (or) higher systems

$$\boxed{e_{ss} = 0}$$

Steady state Error Constants

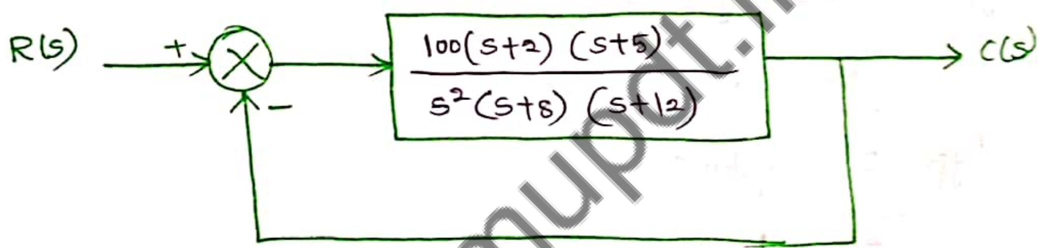
	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Parabolic input $r(t)=\frac{1}{2}t^2$
Type 0 system	$K_p = \lim_{s\to 0} G(s) = K$	0	0
Type-1 system	∞	$K_v = \lim_{s\to 0} sG(s) = K$	0
Type-2 system	∞	∞	$K_a = \lim_{s\to 0} s^2G(s) = K$

Steady State Error (e_{ss})

	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Parabolic Input $r(t)=\frac{1}{2}t^2$
Type-0 System	$\frac{1}{1+K_p}$	∞	∞
Type-1 System	0	$\frac{1}{K_v}$	∞
Type-2 System	0	0	$\frac{1}{K_a}$

Example-1:

For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Sol:

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

Evaluation of Static Error Constants.

Static position Error Constant (K_p)

$$K_p = \lim_{s\to 0} G(s)$$

$$K_p = \lim_{s\to 0}\left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$\boxed{K_p = \infty}$$

Static Velocity Error Constant (K_v)

$$K_v = \lim_{s\to 0} G(s)$$

$$K_v = \lim_{s \to 0} \left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$\boxed{K_v = \infty}$$

Static Acceleration Error Constant (K_a)

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$$K_a = \lim_{s \to 0} \left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \frac{100(0+2)(0+5)}{(0+8)(0+12)}$$

$$\boxed{K_a = 10.4}$$

Steady State Errors

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty}$$

$$\boxed{e_{ss} = 0}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty}$$

$$\boxed{e_{ss} = 0}$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{10.4}$$

$$\boxed{e_{ss} = 0.09}$$

Problem-1: Measurements conducted on a servomechanism show the system response to be $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subjected to a step response. (a) Obtain the expression for the closed loop transfer function. (b) Determine the Undamped Natural frequency and damping ratio.

Sol: Given,

$$\boxed{c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}} \quad (1)$$

Step Response $R(s) = \frac{1}{s}$

Apply Laplace transform for equation (1)

$$L\{c(t)\} = L\{1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}\}$$

$$c(s) = L\{1\} + 0.2\, L\{e^{-60t}\} - 1.2\, L\{e^{-10t}\}$$

$$c(s) = \frac{1}{s} + 0.2\,\frac{1}{s+60} - 1.2\,\frac{1}{s+10}$$

$$c(s) = \frac{(s+60)(s+10) + 0.2(s)(s+10) - 1.2(s)(s+60)}{s(s+60)(s+10)}$$

$$c(s) = \frac{s^2+70s+600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s(s+60)(s+10)}$$

$$\therefore \quad c(s) = \frac{600}{s(s^2+70s+600)}$$

$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}}$$

$\omega_n^2 = 600$

$\boxed{\omega_n = 24.5}$

$2\xi\omega_n = 70$

$2\xi(24.5) = 70$

$\boxed{\xi = 1.428}$

Problem-2: A unity feedback system is characterized by an OLTF $G(s) = \frac{K}{s(s+10)}$. Determine the gain K such that the damping ratio is 0.5. For this value of K determine the time response parameters.

Sol: Given,

OLTF (open loop Transfer function) $G(s) = \frac{K}{s(s+10)}$

closed lood Transfer function (CLTF),

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K}{s^2+10s+K}}$$

Given, damping ratio $(\xi) = 0.5$.

$$2\xi\omega_n = 10$$

$$2(0.5)\omega_n = 10$$

$$\boxed{\omega_n = 10}$$

$$\omega_n^2 = K$$

$$K = (10)^2$$

$$\boxed{K = 100}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{100}{s^2+10s+100}}$$

Peak time, $t_p = \frac{\pi}{\omega_d}$

$$= \frac{3.14}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{3.14}{10\sqrt{1-(0.5)^2}}$$

$$\boxed{t_p = 0.3627}$$

Rise time, $t_r = \dfrac{\pi - \cos^{-1}\xi}{\omega_d}$

$$t_r = \frac{\pi - \cos^{-1}\xi}{\omega_n\sqrt{1-\xi^2}}$$

$$t_r = \frac{3.14 - \cos^{-1}(0.5)}{10\sqrt{1-(0.5)^2}}$$

$$\boxed{t_r = 0.2418}$$

Settling time (2%), $t_s = \dfrac{4}{\xi\omega_n}$

$$t_s = \frac{4}{(0.5)(10)}$$

$$\boxed{t_s = 0.8}$$

Maximum Overshoot, $M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

$$M_p = e^{\frac{-3.14(0.5)}{\sqrt{1-(0.5)^2}}}$$

$$\boxed{M_p = 0.1632 = 16.32\%}$$

Problem-3: The OLTF of a servo system with unity feedback is $G(s) = \dfrac{10}{s(0.1s+1)}$. Evaluate the static error Constants (K_p, K_v, K_a) for the system. Obtain the steady state error of the system when subjected to an input given by the polynomial $r(t) = a_0 + a_1 t + \frac{a_0}{2}t^2$.

Sol) Given,

$$G(s) = \frac{10}{s(0.1s+1)}$$

CLTF $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{10}{0.1s^2 + s + 10}$

$$\therefore \boxed{\frac{C(s)}{R(s)} = \frac{100}{s^2+10s+100}}$$

Given, $\quad r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$

$$R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

Static position Error Constant,

$$K_p = \lim_{s \to 0} G(s) \times a_0$$

$$K_p = \lim_{s \to 0} \frac{10}{s(0.1s+1)} a_0$$

$$K_p = \lim_{s \to 0} \frac{10}{s(0.1s+1)} a_0$$

$$\boxed{K_p = \infty}$$

Static Velocity Error constant,

$$K_v = \lim_{s \to 0} sG(s) \times a_1$$

$$K_v = \lim_{s \to 0} \frac{10s}{s(0.1s+1)} a_1$$

$$\boxed{K_v = 10a_1}$$

Static Acceleration Error Constant,

$$K_a = \lim_{s \to 0} s^2 G(s) \times a_2$$

$$K_a = \lim_{s \to 0} \frac{10s^2}{s(0.1s+1)} a_2$$

$$\boxed{K_a = 0}$$

Steady state errors,

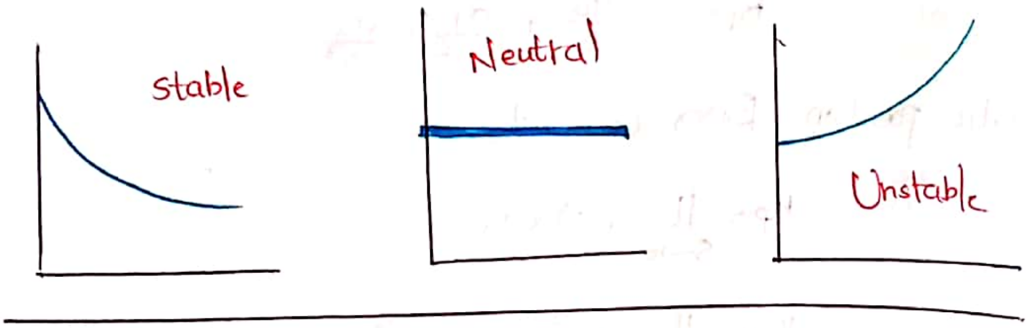
$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0 \quad \text{(step)}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{10a_1} \quad \text{(Ramp)}.$$

$$e_{ss}\text{(parabolic)} = \frac{1}{K_a} = \infty$$

Concept of Stability:

The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



a) Stable b) Neutral c) Unstable

- Roughly, small changes in input, initial conditions (or) system parameters should not result in large changes in the output.
- Very important characteristic of transient response.
- Almost every system designed to be inherently stable.
- Within the boundaries of the parameter variations permitted by stability consideration, we can improve the system performance.
- Two nations of the system stability

  * When the system is excited by a bounded input, the output is bounded.

  * In the absence of an input, the output tends to zero (the equilibrium state of the system) irrespective of the initial conditions, also known as Asympotic stability

- If a system is excited by a unbounded input and produces a unbounded output, nothing can be said about the stability.

- If a system is excited by a bounded input but produces an unbounded output, by definition it is unstable.
- Output of an unstable system may increase to a certain extent and breakdown (or) become non linear so that the linear mathematical model no longer applies.
- For non linear systems, because of possible existence of multiple equilibrium states, it is difficult to define the concept of stability.

Consider a SISO system,

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}} \qquad m<n$$

with initial conditions assumed zero, the output of the system is

$$c(t) = L^{-1}\{G(s)R(s)\} = \int_0^\infty g(T)\,r(t-T)\,dT$$

Taking the absolute values on both sides

$$|c(t)| = \left|\int_0^\infty g(T)\,r(t-T)\,dT\right|$$

Since, absolute value of the integral is not greater than the integrand.

$$|c(t)| \le \int_0^\infty |g(T)\,r(t-T)|\,dT$$

$$|c(t)| \le \int_0^\infty |g(T)|\,|r(t-T)|\,dT$$

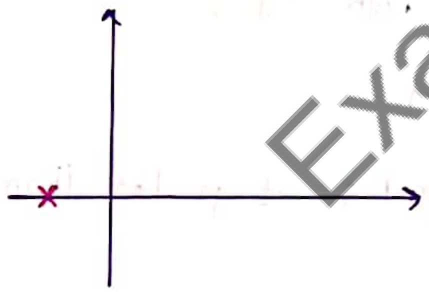
- The BIBO stability is defined as, if every input is bounded $(|r(t)| \le M_1 < \infty)$ then the output is bounded $(c(t) \le M_2 < \infty)$
- From above equation,

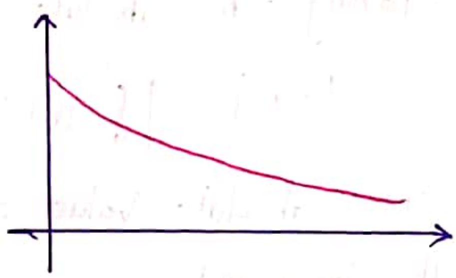
$$|c(t)| \le M_1 \int_0^\infty |g(T)|\,dT \le M_2$$

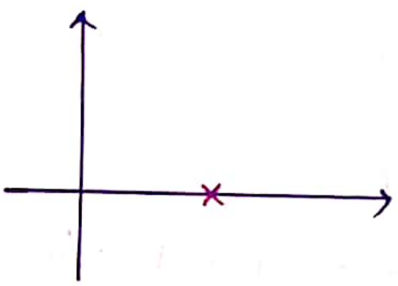
- Thus, by BIBO Stability Condition is Satisfied if the impulse response is absolutely integrable and finite
- The nature of g(t) is dependent on the poles of the transfer function G(s) which are the roots of the characteristic equation.
- The roots could be real, Complex Conjugate and may have multiplicity of Various orders.
- The nature of the response terms Contributed by all possible types of root is as below.

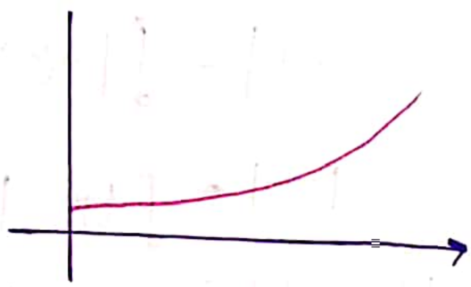
Response terms Contributed by Various types of roots

Type of Root	Nature of response terms Contributed
Single root at $s = \pm\sigma$	$Ae^{\pm\sigma t}$
Roots of multiplicity K at $s = \pm\sigma$	$(A_1 + A_2 t + A_3 t^2 + \cdots + A_K t^{K-1})\, e^{\pm\sigma t}$

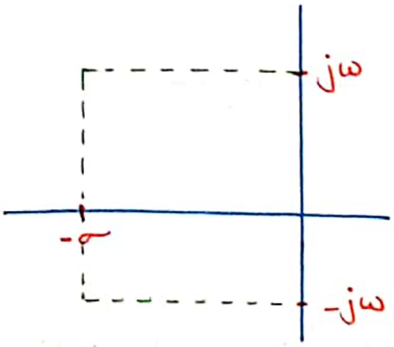


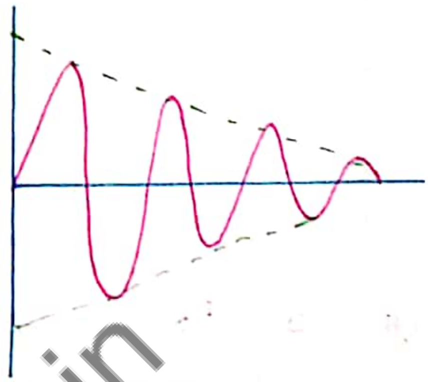


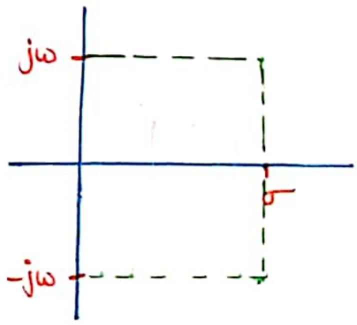


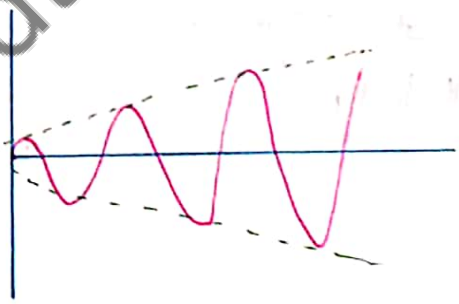


Type of Root	Nature of response terms Contributed
Complex Conjugate roots at $s=\sigma\pm j\omega$	$Ae^{at}\sin(\omega t+\beta)$
Complex conjugate Root pairs of multiplicity K at $s=\sigma\pm j\omega$	$[A_1\sin(\omega t+\beta_1)+A_2 t\sin(\omega t+\beta_2)+A_3t^2\sin(\omega t+\beta_3)+\cdots\cdots+A_K t^{K-1}\sin(\omega t+\beta_K)]\,e^{\sigma t}$

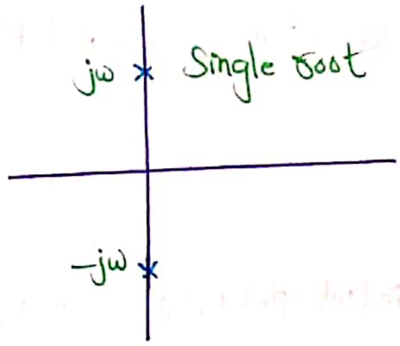


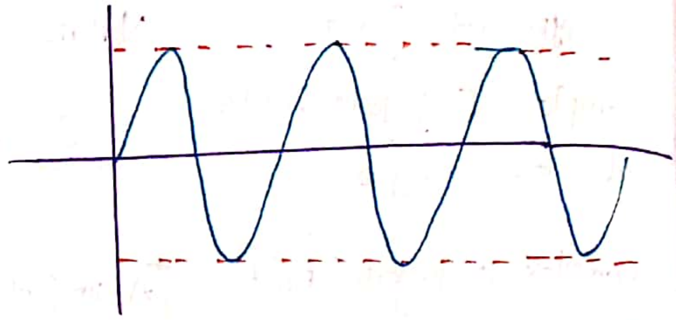


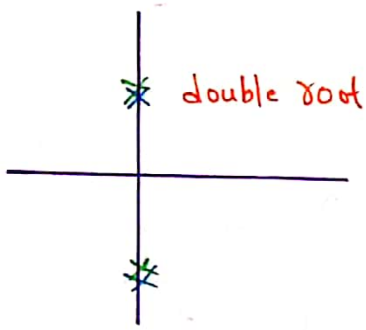


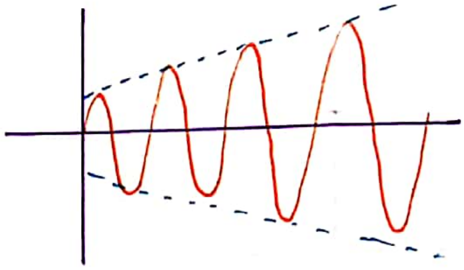


Type of Root	Nature of Response terms Contributed
Simple Complex Conjugate roots on jω axis at $s=\pm j\omega$	$A\sin(\omega t+\beta)$
Complex Conjugate Root Pairs of multiplicity K on jω axis at $s=\pm j\omega$	$A_1\sin(\omega t+\beta_1)+A_2t\sin(\omega t+\beta_2)+A_3t^2\sin(\omega t+\beta_3)+A_4t^3\sin(\omega t+\beta_4)+\cdots\cdots+A_Kt^{K-1}\sin(\omega t+\beta_K)$

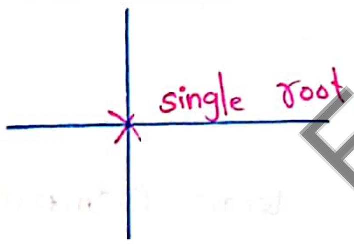


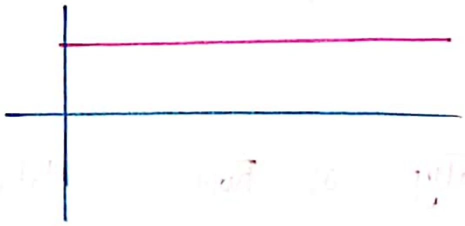


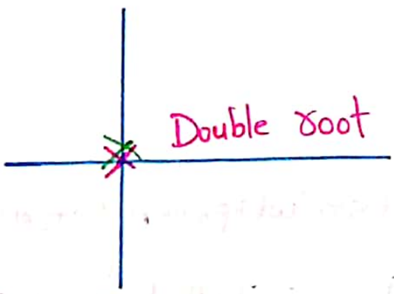


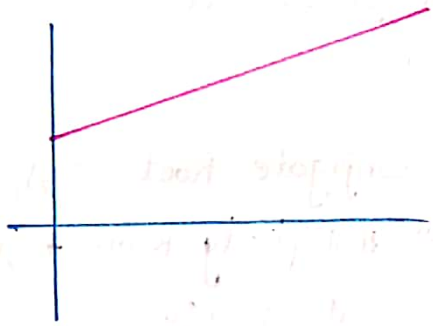


Type of Root	Nature of response terms Contributed
Single root at origin s=0	A
Pole at origin of multiplicity K	$A_1 + A_2 t + A_3 t^2 + \cdots + A_k t^{K-1}$

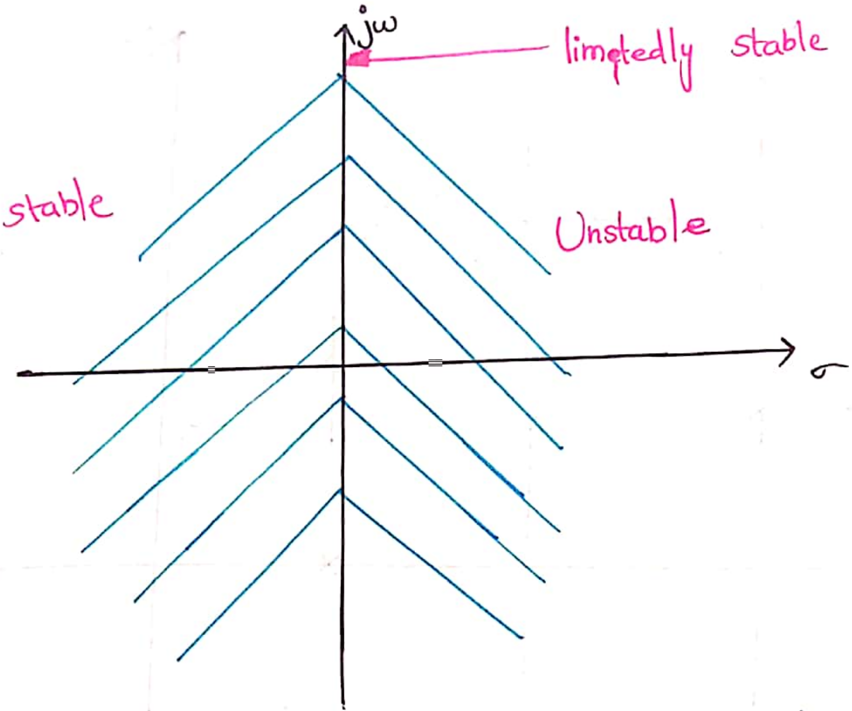






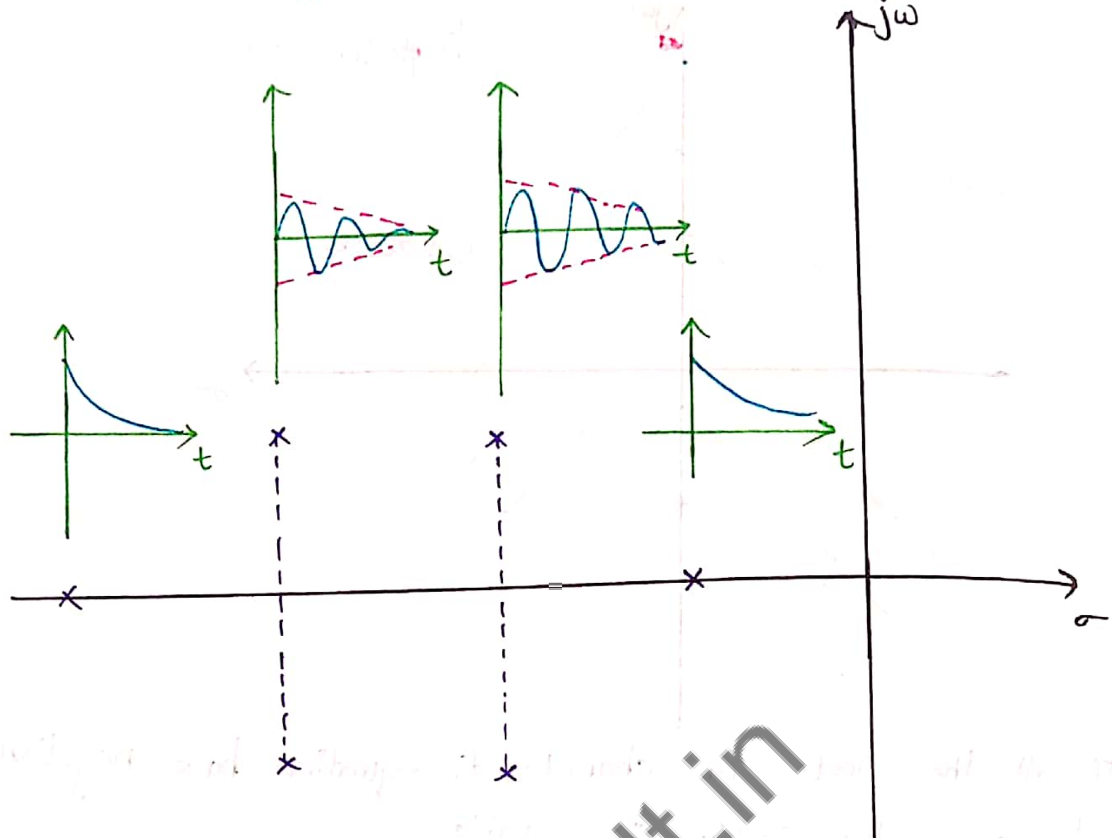


Conclusion on stability based on root location



- If all the roots of characteristic equation have negative real parts, the system is stable
- If any root of the characteristic equation has a positive real part (or) if there is a repeated pole on the jω-axis, the system is unstable
- If the first condition is satisfied, but for a non repeated root on the jω-axis, the system is limitedly stable.
- Futher sub-division of the concept of stability, a linear system is characterized as
  - * Absolutely stable with respect to a parameter of the system if it is stable for all values of this parameter.
  - * Conditionally stable with respect to a parameter, if the system is stable for only certain bounded ranges of values of this parameter.

Relative Stability



- Not Sufficient to know whether system is stable or not but meet the requirements of specifications of relative stability
- Relative stability is quantitative measure of how fast the transients die out or the system attain settling condition.
- Settling time of a pair of complex roots is inversely proportional to the real part (negative sign) of the roots
- As a pair of roots moves farther away from the imaginary axis, the relative stability of the system increases

Necessary Condition for the stability

Inspection test:

A necessary but not sufficient condition for stability of a linear system is that all the coefficients of the characteristic equation $q(s) = 0$ be real and have the same sign. Futhermore, none of the coefficients must be zero.

Consider the characteristic equation,

$$q(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \quad ; \; a_0 > 0$$

The above equation can be rewritten as

$$q(s) = a_0 \prod (s + s_K) \prod \left[(s + \sigma_1)^2 + \omega_1^2 \right]$$

- For the system to be stable the roots should have negative real part, which is satisfied only if all s_K and σ_1 are positive real i.e. all the factors in the above equation have positive terms only
- As these factors are multiplied together for the characteristic equation, all the coefficients are of the characteristic polynomial must be positive.
- None of the coefficients must be zero (or) negative except:
  - * One (or) more roots have positive real parts.
  - * A root (roots) at origin i.e $s_K = 0$ and hence $a_n = 0$
  - * $\sigma_1 = 0$ for some 1, which implies the presence of the roots on the $j\omega$ axis.
- Absence (or) negativeness of any of the coefficients of the characteristics equation indicates that the system is either unstable (or) at most marginally stable.
- positiveness of the coefficients of the characteristic equation ensures the negativeness of the real roots but does not ensure the negativeness of the real parts of the complex roots. Therefore it cannot be sufficient condition for stability of higher order systems.
- After the inspection test (or) necessary condition is satisfied one should proceed to examine the sufficient conditions for stability.

- A. Hurwitz and E. J Routh independently published the method for the sufficient condition of stability.
- Hurwitz criterion is in terms of determinants and Routh's criterion is in terms of array formulation. The combination called the Routh Hurwitz (R-H) criterion is the sufficient condition for stability

Constructing the Routh Array

Consider the characteristic equation described by

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

- Only the first 2 rows of the array are obtained from the characteristic equation the remaining are calculated as follows.

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & \cdots\ a_n \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
\vdots & & & & \\
s^2 & e_1 & e_2 & & \\
s^1 & f_1 & & & \\
s^0 & a_n & & &
\end{array}$$

Coefficients b_1, b_2 are evaluated as follows

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \quad \text{and so on.}$$

- The process is continued till we get a zero as the last coefficient in the third row.
- The coefficients of the 4^{th}, 5^{th}, n^{th} and $(n+1)^{th}$ row are evaluated as:

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

- In the process of generating the Routh array, the missing terms are regarded as zero.

Routh Hurwitz Stability Criterion

- It is a method of determining continuous system stability
- The Routh-Hurwitz criterion states that "the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array".
- This method yields stability information without the need to solve for the closed-loop system poles.
- Using this method,

  We can tell how many closed-loop system poles are in the left half-plane, in the right half-plane & on the jw-axis.
- The method requires two steps:

  * Generate a data table called a Routh table.

* Interpret the Routh table to tell how many closed-loop system poles are in the LHP, the RHP and on the $j\omega$-axis.

Routh's Stability Criterion.

- If the closed-loop transfer function has all poles in the left half of the s-plane, the system is Stable.
- Thus, a system is stable if there are no sign changes in the first Column of the Routh table.
- The Routh-Hurwitz Criterion declares that the number of roots of the polynomial that lies in the right half-plane is equal to the number of sign changes in the first Column.
- Hence the system is unstable if the poles lies on the right hand side of the s-plane.

Example-1:

The forward path transfer function of a Unity Feedback System is given by $G(s) = \frac{K}{(s+2)(s+3)(s+4)(s+5)}$. Assess the stability of the system when i) $K=100$ and ii) $K=1000$

Sol:

$$1+G(s)H(s) = 0$$

$$1+\frac{K}{(s+2)(s+3)(s+4)(s+5)} = 0$$

$$(s^2+5s+6)(s^2+9s+20)+K=0$$

$$s^4+9s^3+20s^2+5s^3+45s^2+100s+6s^2+54s+120+K=0$$

$$s^4+14s^3+71s^2+154s+120+K=0$$

When $K = 100$

s^4	1	71	220
s^3	14	154	0
s^2	60	220	0
s^1	102.667	0	
s^0	220		

$$\begin{vmatrix} 1 & 71 \\ 14 & 154 \end{vmatrix} = \frac{(14\times 71) - (1\times 154)}{14} = 60$$

$$\begin{vmatrix} 1 & 220 \\ 14 & 0 \end{vmatrix} = \frac{(14\times 220) - (1\times 0)}{14} = 220$$

$$\begin{vmatrix} 14 & 154 \\ 60 & 220 \end{vmatrix} = \frac{(60\times 154) - (14\times 220)}{60} = 102.667$$

Absolutely stable.

When $K = 1000$; $s^4 + 14s^3 + 71s^2 + 154s + 1120 = 0$

s^4	1	71	1120
s^3	14	154	
s^2	60	1120	
s^1	-107.3		
s^0	1120		

$$\begin{vmatrix} 1 & 71 \\ 14 & 154 \end{vmatrix} = \frac{(14\times 71) - (1\times 154)}{14} = 60$$

$$\begin{vmatrix} 1 & 1120 \\ 14 & 0 \end{vmatrix} = \frac{(14\times 1120)}{14} = 1120 \; ; \; \begin{vmatrix} 14 & 154 \\ 60 & 1120 \end{vmatrix} = -107.3$$

- Two sign changes.
- Two roots lie in Right Half of s-plane.

s^4	1	71	120+K
s^3	14	154	0
s^2	60	120+K	
s^1	$126-\frac{7K}{30}$		
s^0	120+K		

$$\begin{vmatrix} 14 & 154 \\ 60 & 120+K \end{vmatrix} = \frac{(60\times 154)-(14\times(120+K))}{60} = 126-\frac{7K}{30}$$

$126-\frac{7K}{30} > 0$ and $120+K>0$

$\boxed{K > -120}$

$\frac{7K}{30} < 126$

$7K < \cancel{37}80$

$\boxed{K < 540}$

$\boxed{-120 < K < 540}$

Example-2:

The forward path transfer function of a unity feedback system is given by $G(s) = \frac{5k}{s^3+9s^2+13s+k}$. Find the range of K such that the System is stable

Sol:

$1+G(s)H(s) = 0$

$1+\frac{5K}{s^3+9s^2+13s+K} = 0$

$s^3+9s^2+13s+6K=0$

s^3	1	13
s^2	9	6K
s^1	$\frac{117-6K}{9}$	0
s^0	6K	

Condition - 1 : $\frac{117-6K}{9} > 0$

Condition - 2 : $6K > 0$

$\boxed{K > 0}$

$\frac{117-6K}{9} > 0$

$6K < 117$

$\boxed{K < 19.5}$

$\boxed{0 < K < 19.5}$

Stable

Example-3 :

Assess the stability of the system described by the characteristic equation of a system is described by $s^5+2s^4+2s^3+9s^2+11s+10=0$

Sol: Given,

$$s^5+2s^4+2s^3+9s^2+11s+10=0.$$

s^5	1	2	11
s^4	2	9	10
s^3	$\frac{-5}{2}$	6	
s^2	$\frac{69}{5}$	10	
s^1	$\frac{539}{69}$		
s^0	10		

$$\begin{vmatrix} 2 & 9 \\ -\frac{5}{2} & 6 \end{vmatrix} = \frac{69}{5}$$

$$\begin{vmatrix} -\frac{5}{2} & 6 \\ \frac{69}{5} & 10 \end{vmatrix} = \frac{539}{60}$$

- Two sign changes and these 2 roots lies in the Right Half plane (RHP)

Example-4:

Asses the stability of the system with characteristic equation $s^5+2s^4+2s^3+9s^2+10s+20=0$ using Routh-Hurwitz Criterion.

Ans: Given,

$$s^5+2s^4+2s^3+9s^2+10s+20=0$$

s^5	1	2	10
s^4	2	9	20
s^3	$-\frac{5}{2}$	0	
s^2	9	20	
s^1	$\frac{50}{9}$		
s^0	0		

- Two sign changes and these two roots lie in the Right Half Plane (RHP).

Example-5:

A Unity Feedback system with $G(s) = \dfrac{K}{s(0.4s+1)(0.25s+1)}$.

Using the Routh criterion find the range of K for stability, marginal value of K and the frequency of sustained oscillations.

Sol:

$$1+G(s)H(s) = 0$$

$$1+\frac{K}{s(0.4s+1)(0.25s+1)} = 0$$

$$s(0.4s+1)(0.25s+1)+K=0$$

$$s(0.1s^2+0.65s+1)+K=0$$

$$0.1s^3+0.65s^2+s+K=0$$

$$s^3+6.5s^2+10s+10K=0$$

s^3	1	10
s^2	6.5	10K
s^1	$\frac{65-10K}{6.5}$	
s^0	10K	

$\frac{65-10K}{6.5} > 0$ and $10K > 0$

$\boxed{K > 0}$

$65 - 10K > 0$

$10K < 65$

$\boxed{K < 6.5}$

$\boxed{0 < K < 6.5}$

Let $K = 6.5$

$s^3 + 6.5s^2 + 10s + 65 = 0$

$s^2(s+6.5) + 10(s+6.5) = 0$

$(s+6.5)(s^2+10) = 0$

$\boxed{s = -6.5}$ $\boxed{s = \pm j\sqrt{10}}$

Frequency of Sustained oscillations,

$\boxed{\omega = \sqrt{10}}$

1) Determine the stability of the systems — Homework.

i) $s^4 + 2s^3 + 8s^2 + 4s + 3 = 0$

Ans:

s^4	1	8	3
s^3	2	4	
s^2	6	3	
s^1	3		
s^0	3		

$\therefore$ It is absolutely stable.

ii) $s^5+s^4+3s^3+9s^2+16s+10=0$

s^5	1	3	16
s^4	1	9	10
s^3	-6	6	
s^2	10	10	
s^1	12		
s^0	10		

- Two sign changes and these 2 roots lies in the Right Half plane (RHP)

iii) $s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	0	0	0	
s^3	8	12		
s^2	3	4		
s^1	$\frac{4}{3}$	0		
s^0	4			

Auxiliary polynomial is always even

$$A[s] = 2s^4+6s^2+4$$

$$\frac{d}{ds}A[s] = 8s^3+12s+0.$$

2. Determine range of K for the system to be stable

i) $s^3+2Ks^2+(K+2)s+4=0$

Sol:- Given $s^3+2Ks^2+(K+2)s+4=0.$

s^3	1	$K+2$
s^2	$2K$	4
s^1	$\frac{K^2+2K-2}{K}$	0
s^0	4	

$$\begin{vmatrix} 1 & K+2 \\ 2K & 4 \end{vmatrix} = \frac{2K(K+2)-4}{2K} = \frac{K^2+2K-2}{4}$$

$2K>0$ $\qquad$ $K+2>0$ $\qquad$ $\frac{K^2+2K-2}{K}>0$

$K>0$ $\qquad$ $K>-2$

Not stable.

$K^2+2K-2>0$

$= \frac{-2 \pm \sqrt{4-4(1)(-2)}}{2}$

$K<-1-\sqrt{3}$; $-1+\sqrt{3}<K$

$K<-2.732$

Not stable

$-1+\sqrt{3}<K$

$K>0.7320$

$\therefore$ $\boxed{0<K<0.732}$

ii) $s^4+4s^3+13s^2+36s+K=0$

s^4	1	13	K
s^3	4	36	
s^2	4	K	
s^1	$36-K$		
s^0	K		

$$\begin{vmatrix} 4 & 36 \\ 4 & K \end{vmatrix} = \frac{(4\times 36)-(4\times K)}{4} = \frac{4(36-K)}{4} = 36-K$$

$K>0$ $\qquad$ $36-K>0$

$K<36$

$\boxed{0<K<36}$ stable

iii) $s^4 + 20Ks^3 + 5s^2 + 10s + 15 = 0$

$$
\begin{array}{c|ccc}
s^4 & 1 & 5 & 15 \\
s^3 & 20K & 10 & \\
s^2 & \dfrac{10K-1}{2K} & 15 & \\
s^1 & \dfrac{100K-10-600K^2}{10K-1} & 0 & \\
s^0 & 15 & &
\end{array}
$$

$$\begin{vmatrix} 1 & 5 \\ 20K & 10 \end{vmatrix} = \frac{20K(5) - 1\times 10}{20K} = \frac{10(10K-1)}{20K} = 10K-1$$

$$\begin{vmatrix} 20K & 10 \\ \frac{10K-1}{2K} & 15 \end{vmatrix} = \frac{10\left(\frac{10K-1}{2K}\right) - 20K(15)}{\frac{10K-1}{2K}} = \frac{100K - 10 - (20\times 15\times K\times 2K)}{10K-1}$$

$$= \frac{100K-10-600K^2}{10K-1}$$

$20K > 0$

$K > 0$

$\frac{10K-1}{2K} > 0$

$10K - 1 > 0$

$10K > 1$

$K > \frac{1}{10}$

$K > 0.1$

$\frac{100K-10-600K^2}{10K-1} > 0$

$100K - 10 - 600K^2 > 0$

$600K^2 - 100K + 10 < 0$

$$\frac{100 \pm \sqrt{(100)^2 - 4(600)(10)}}{2(600)}$$

Example-5:

Assess the stability of the system described by the characteristic equation of a system is described by $s^5+2s^4+2s^3+4s^2+11s+10=0$

Sol:

s^5	1	2	11	+ve
s^4	2	4	10	+ve
s^3	$\boxed{0}^{\epsilon}$	6		0
s^2	$\frac{4\epsilon-12}{\epsilon}$	10		-ve
s^1	$\frac{-10\epsilon^2+24\epsilon-72}{4\epsilon-12}$			+ve
s^0	10			+

2 sign changes.

First element zero then replace zero with 'ϵ' (ephlson)

Two sign changes and 2 roots lie in Right Half plane (RHP)

When first element of the row is zero, it leads to division by '0' in next rows. so replace '0' with 'ϵ' and complete the array. The system is anyway unstable.

Example-6:

Assess the stability of the system with characteristic equation $s^5+2s^4+2s^3+4s^2+10s+20=0$ using Routh Hurwitz criterion.

Sol:

s^5	1_{s^5}	2_{s^3}	10_{s^1}
s^4	2_{s^4}	4_{s^2}	20_{s^0}
s^3	$\boxed{0}$ 8	$\boxed{0}$ 8	
s^3	1	1	
s^2	2	20	
s^1	-9		
s^0	20		

A complete row of zero's then we consider Auxiliary equation.

Auxiliary equation,

$$A[s] = 2s^4 + 4s^2 + 20$$

$$\frac{d}{ds} A[s] = 8s^3 + 8s + 0$$

Two sign changes and 2 roots in Right Half plane

Auxiliary equation has root on $j\omega$ axis (or) common factor.

$$A[s^3] = 2s^4 + 4s^2 + 20$$

$$2s^4+4s^2+20 \overline{)\, s^5 + 2s^4 + 2s^3 + 4s^2 + 10s + 20 \,} \left(\tfrac{1}{2}s + 1\right.$$

$$s^5 + \quad 2s^3 + \quad 10s$$

$$2s^4 + 4s^2 + 20$$

$$2s^4 + 4s^2 + 20$$

$$0$$

$$\therefore s^5 + 2s^4 + 2s^3 + 4s^2 + 10s + 20 = (s+2)(s^4 + 2s^2 + 10)$$

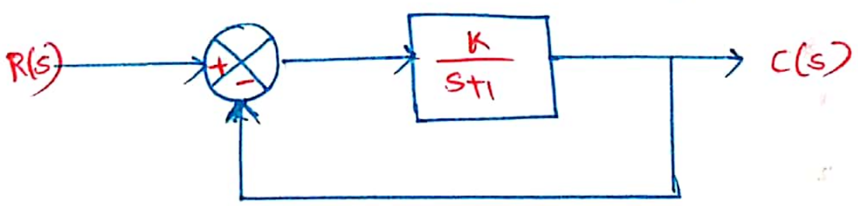
Roots are $\pm\sqrt{3}, \pm j2\sqrt{2}$

Marginal roots.

Outline:

Introduction:

Consider a Unity feedback Control system shown below.



The open loop transfer function $G(s)$ of the system is

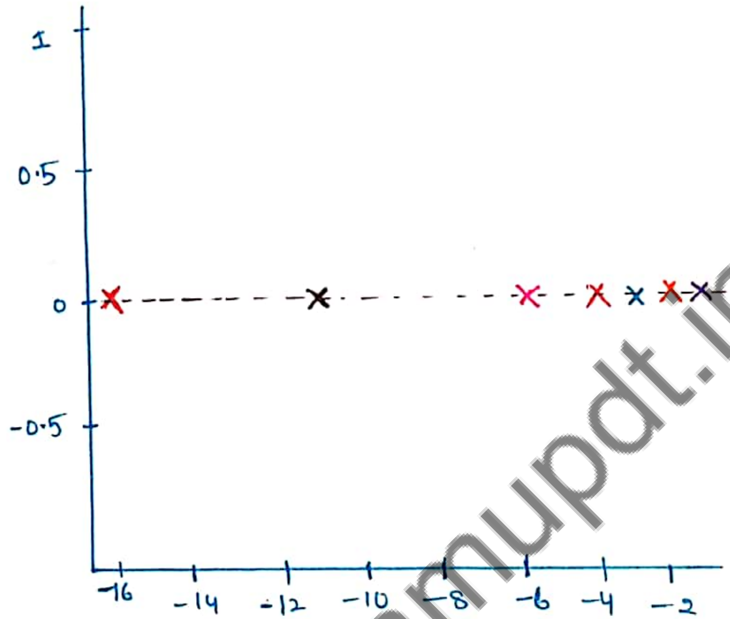
$$G(s) = \frac{K}{s+1}$$

And the closed transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s+1+K}$$

Location of closed loop pole for different values of K. (remember K>0).

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$



K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16

Rout Locus:

The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

How to Sketch Root locus?

One way is to compute the roots of the characteristic equation for all possible values of K.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16

Computing the roots for all values of K might be tedious for higher order systems.

$$\boxed{\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+10)(s+20)+K}}$$

K	pole
0.5	?
1	?
2	?
3	?
5	?
10	?
15	?

Construction of Root Loci

- Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution.
- A simple method for finding the roots of the characteristic

equation has been developed by W.R. Evans and used extensively in Control engineering.

- This method, called the root-locus method, is one in which the roots of the characteristic equation are plotted for all Values of a system parameter.
- The roots Corresponding to a particular Value of this parameter Can then be located on the resulting graph.
- By using the root-locus method the designer can predict the effects on the location of the closed-loop Poles of Varying the gain Value (or) adding open-loop poles and/or open-loop zeros.
- The characteristic equation is obtained by setting the denominator polynomial equal to zero

$$\boxed{1+G(s)H(s)=0}$$

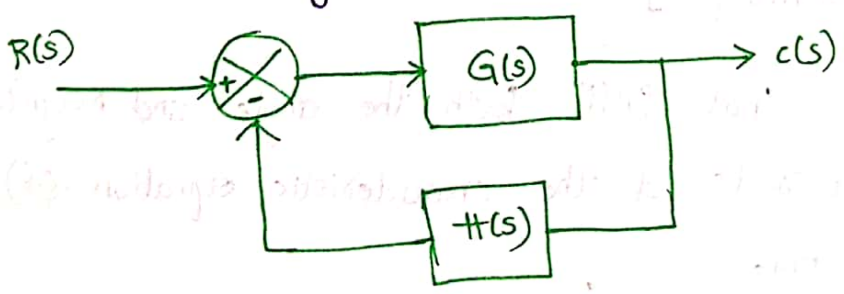
(or)

$$\boxed{G(s)H(s)=-1}$$

Since, $G(s)H(s)$ is a complex quantity it can be split into angle and magnitude part.

Angle and Magnitude Conditions

- In Constructing the root loci angle and magnitude Conditions are important.
- Consider the system shown in the figure.



The closed loop transfer function is

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}}$$

The characteristic equation is obtained by setting the denominator polynomial equal to zero.

$$\boxed{1+G(s)H(s) = 0}$$

(or)

$$\boxed{G(s)H(s) = -1}$$

Since $G(s)H(s)$ is a complex quantity. It can be split into angle and magnitude part.

The angle of $G(s)H(s) = -1$ is

$$\angle G(s)H(s) = \angle -1$$

$$\boxed{\angle G(s)H(s) = \pm 180^\circ (2K+1)}$$

Where $K = 1, 2, 3 \ldots$

The magnitude of $G(s)H(s) = -1$ is

$$|G(s)H(s)| = |-1|$$

$$\boxed{G(s)H(s) = 1}$$

Angle Condition

$$\boxed{\angle G(s)H(s) = \pm 180^\circ (2K+1)}$$

$K = 1, 2, 3 \ldots$

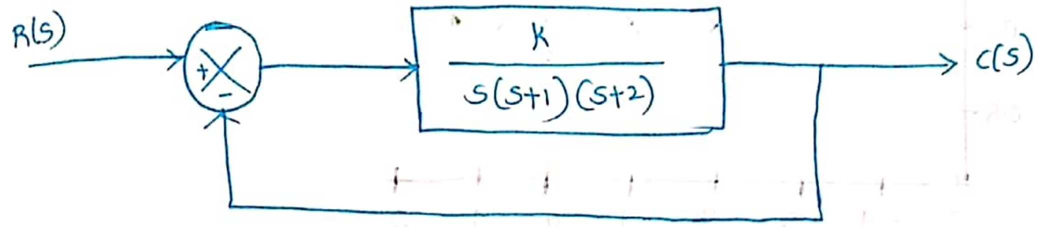
Magnitude Condition

$$\boxed{|G(s)H(s)| = 1}$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation (or) the closed-loop poles.

Construction of root loci

Step-1: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.

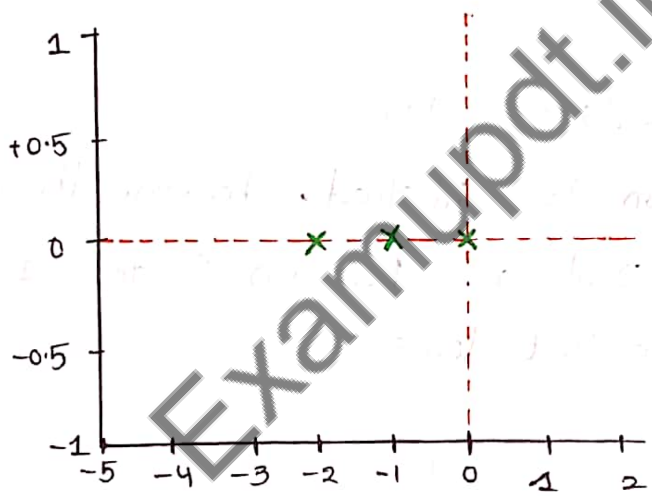


$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

The angle is $-\angle s - \angle s+1 - \angle s+2$

$s(s+1)(s+2)$; $s=0$; $s=-1$, $s=-2$

Poles represents with ×, Zeros represents with o.



Step-2: Determine the root loci on the real axis

- To determine the root loci on real axis we select some test points.

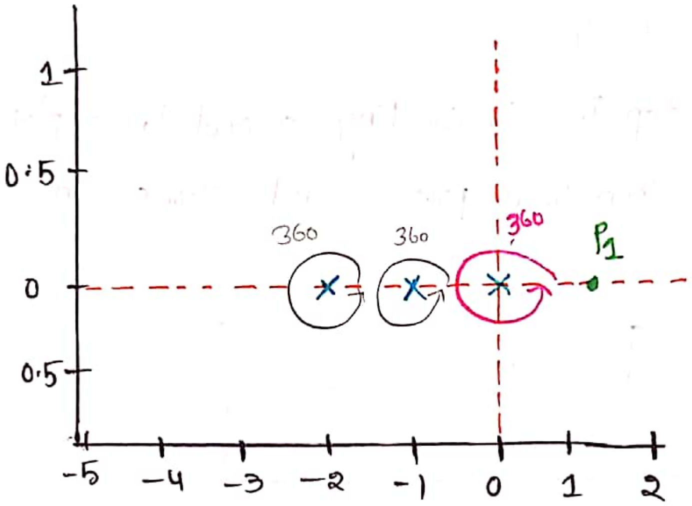
Eg: P_1 (on positive real axis).

$$\angle s = \angle s+1 = \angle s+2 = 0^\circ$$

The angle condition is not satisfied.

Hence, there is no root locus on the positive real axis.

$\boxed{360 = 0^\circ}$



$$360 + 360 + 360 = 1080 = 180(6) = 180^\circ(2(3)+0)$$

Next, select a test point on the negative real axis between 0 and -1.

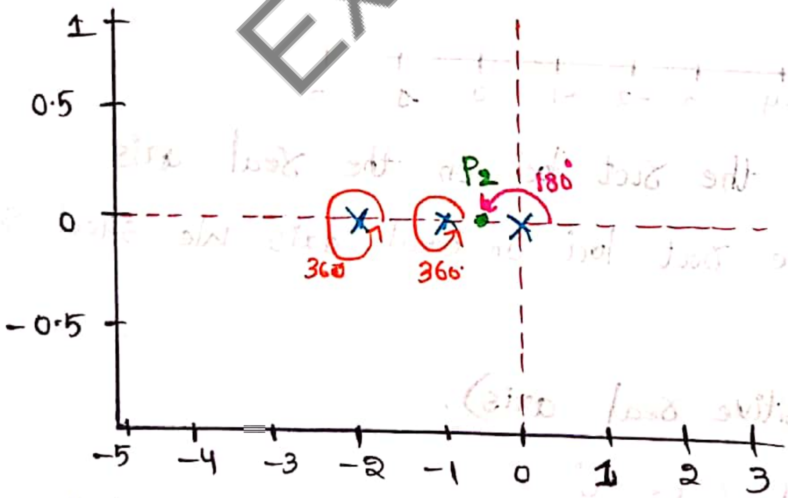
Then,

$$\angle s = 180^\circ, \quad \angle s+1 = \angle s+2 = 0^\circ \qquad [\text{For } P_2]$$

Thus,

$$-\angle s - \angle s+1 - \angle s+2 = -180^\circ$$

The angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and -1 forms a portion of the root locus.



Now, select a test point on the negative real axis between -1 and -2

Then,

$$\angle s = \angle s+1 = 180^\circ, \qquad \angle s+2 = 0$$

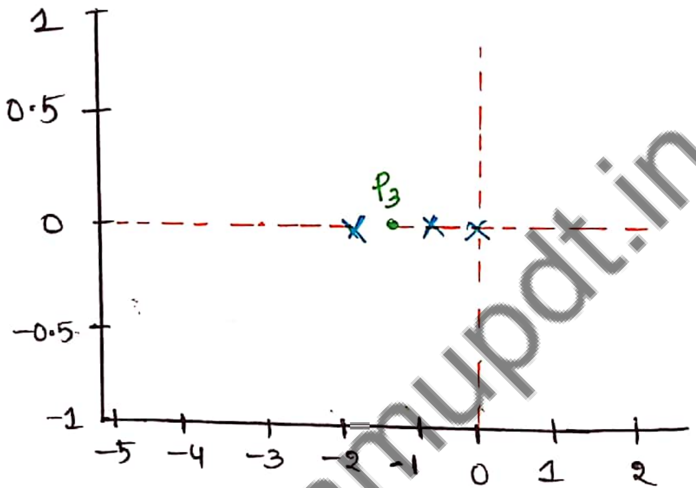
Thus,

$$-\angle s - \angle s+1 - \angle s+2 = -360^\circ$$

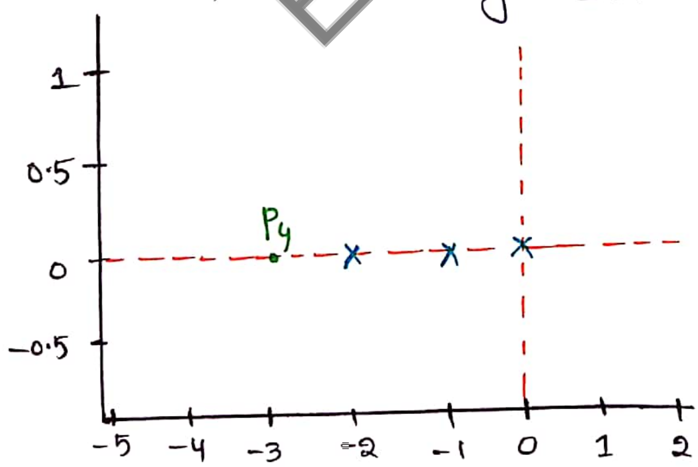
The angle condition is not satisfied. Therefore, the negative real axis between -1 and -2 is not a part of the root locus.

If angle is 180 (even number) then root doesn't exists in locus in +ve real axis.

If angle is 180 (odd number) then root exists in locus in +ve real axis.

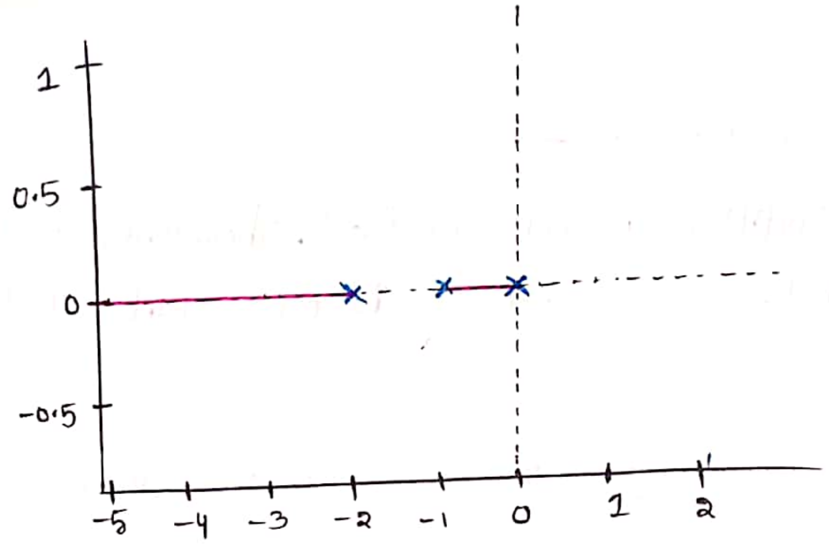


Simalarly, test point on the negative real axis between -2 and $-\infty$ satisfies the angle condition.



Therefore, the negative real axis between -2 and $-\infty$ is part of the root locus.

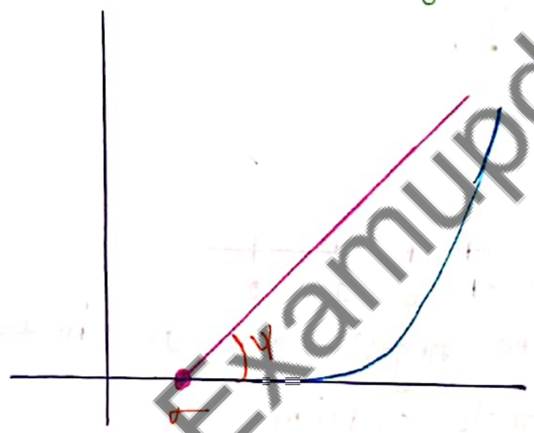
On the real axis, if any segment has odd number of roots lie to the right side and is part of root locus. and if even not part of root locus.



- Pink line is a part of root locus.
- Breakpoint lies in root locus (pink line)

Step-3: Determine the asymptotes of the root loci. That is, the root loci when s is far away from origin

Asymptote is the straight line approximation of a curve



— Asymptotic Approximation
— Actual Curve

$\sigma \rightarrow$ Centroid of Asymptotes

$\psi \rightarrow$ Angle of Asymptotes.

n – poles
n - Segment of root locus
m – zero's
m – Segment will end at open loop zero's

$(n-m) \longrightarrow$ Infinite zero.

$$\boxed{\text{Angle of asymptotes} = \psi = \frac{\pm 180^\circ(2K+1)}{n-m}}$$

Where,

$n \longrightarrow$ number of poles

$m \longrightarrow$ number of zeros

For this Transfer function,

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$\frac{K}{s(s+1)(s+2)}$ approaches $\frac{K}{(s+1)^3}$ When s is large,

Angle condition is,

$$-3\angle s+1 = \pm 180^\circ(2K+1)$$

$$\boxed{\psi = \frac{\pm 180^\circ(2K+1)}{3-0}}$$

$\psi = \pm 60^\circ$ When $K=0$

$\psi = \pm 180^\circ$ When $K=1$

$\psi = \pm 300^\circ$ When $K=2$

$\psi = \pm 420^\circ$ When $K=3$

$K = 0, 1, 2, 3 \ldots\ldots (n-m-1)$

- Since, the angle repeats itself as K is varied, the distinct angles for the asymptotes are determined as, 60°, -60° and 180°

- Before we can draw these asymptotes in the complex plane, we need to find the point where they intersect the real axis.

- Point of intersection of asymptotes on real axis is

Real axis (or) Centroid of asymptotes

$$\boxed{\sigma = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{n-m}}$$

For $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$\dfrac{K}{s(s+1)(s+2)}$ approaches $\dfrac{K}{(s+1)^3}$ When s is large

Angle condition is

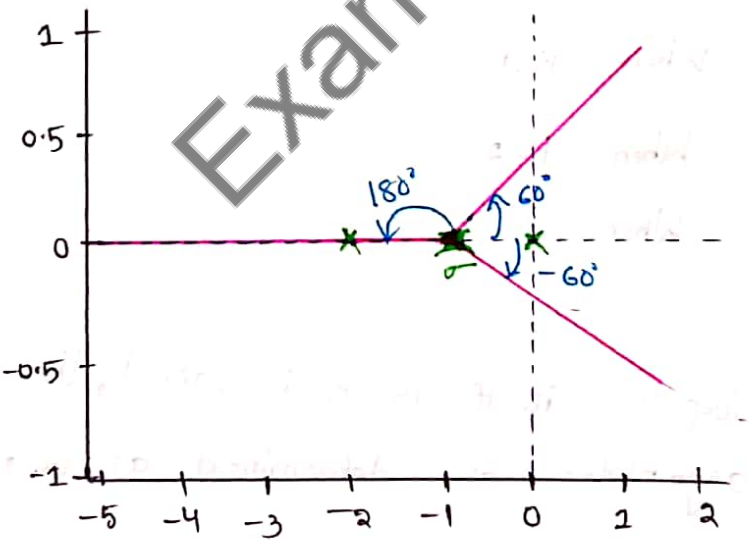
$$-3\angle s+1 = \pm 180^\circ(2K+1)$$

$$\sigma = \frac{(0-1-2)-0}{3-0}$$

$$\sigma = \frac{-3}{3}$$

$$\boxed{\sigma = -1}$$

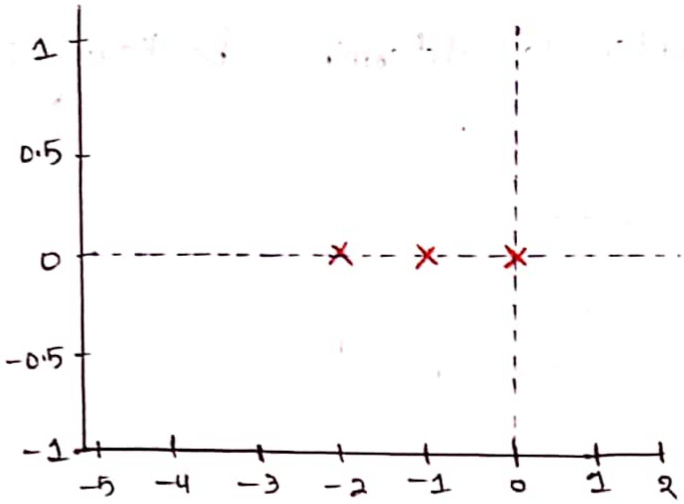
$\psi = 60^\circ, -60^\circ, 180^\circ$ and $\sigma = -1$



We are plotting angles at -1.

Step-4: Determine the break away | break-in point.

The breakaway / break-in point is the point from which the root locus branches leaves / arrives the real axis



- The breakaway (or) break-in points can be determined from the roots of

$$\boxed{\frac{dK}{ds} = 0}$$

- It should be noted that not all the solutions of $\frac{dK}{ds} = 0$ correspond to actual breakaway points.
- If a point at which $\frac{dK}{ds} = 0$ is on a root locus, it is an actual breakaway (or) break-in point.

The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$\boxed{K = -[s(s+1)(s+2)]}$$

The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

$$\frac{dK}{ds} = \frac{-d}{ds}\left[s^3+3s^2+2s\right]$$

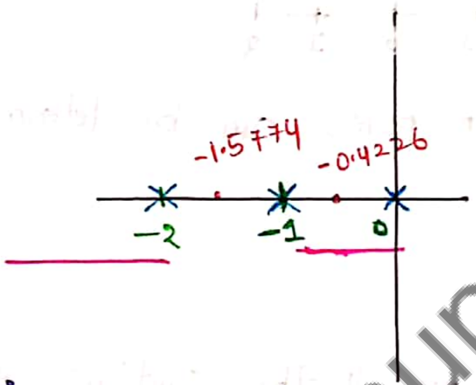
$$\boxed{\frac{dK}{ds} = -3s^2-6s-2}$$

Set $\frac{dK}{ds}=0$ in order to determine breakaway point.

$$-3s^2-6s-2=0$$

$$3s^2+6s+2=0$$

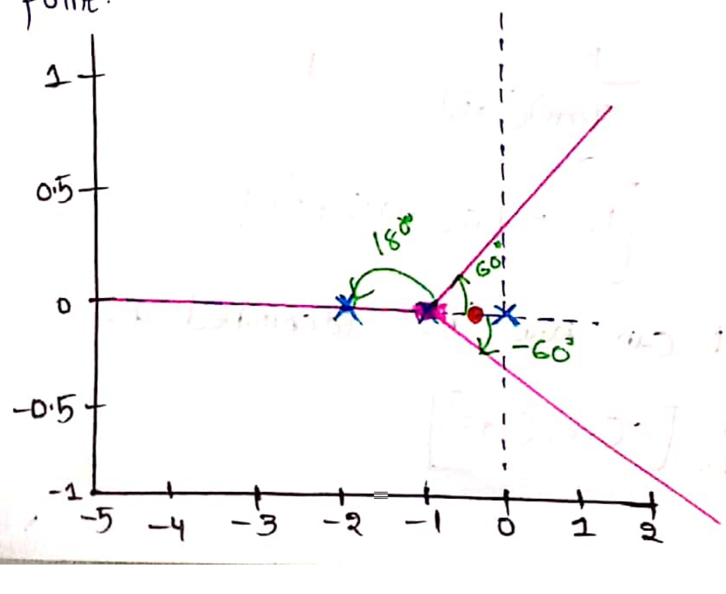
$$\boxed{\begin{aligned} s &= -0.4226 \\ s &= -1.5774 \end{aligned}}$$



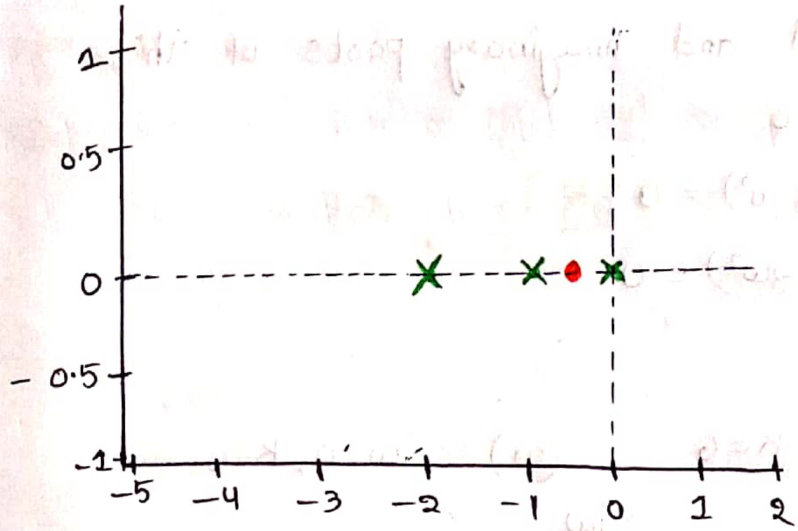
Pink line area in root locus.

Breaking point exists in root locus i.e b/w -1 & 0

- Since the breakaway point needs to be on root locus between 0 and -1, it is clear that $s=-0.4226$ corresponds to the actual breakaway point.
- Point $s=-1.5774$ is not on the root locus. Hence the point is not an actual breakaway (or) break-in point.

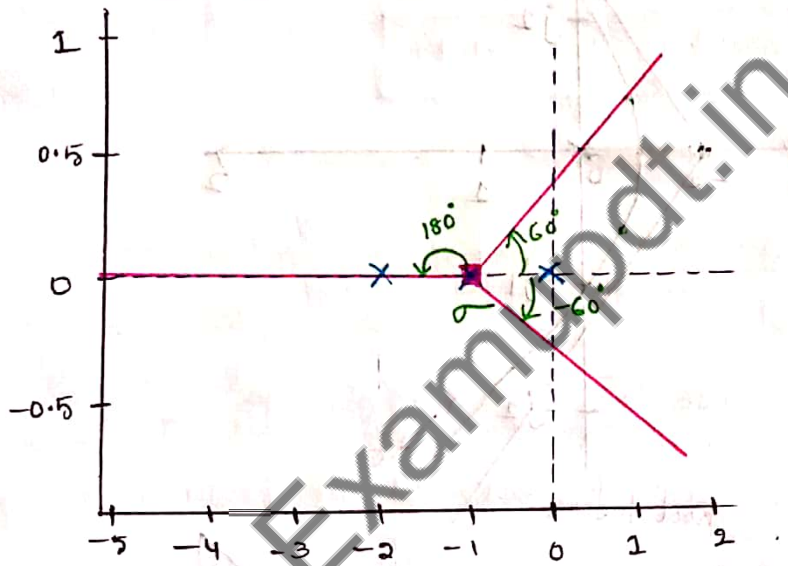


$s=-0.4226$



$S = -0.4226$

Step-5: Determine the points where root loci across the imaginary axis



- Let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero and then solve for ω and K.

For present system the characteristic equation is

$$s^3 + 3s^2 + 2s + K = 0$$

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

$$(K - 3\omega)^2 \rightarrow \text{Real part}$$

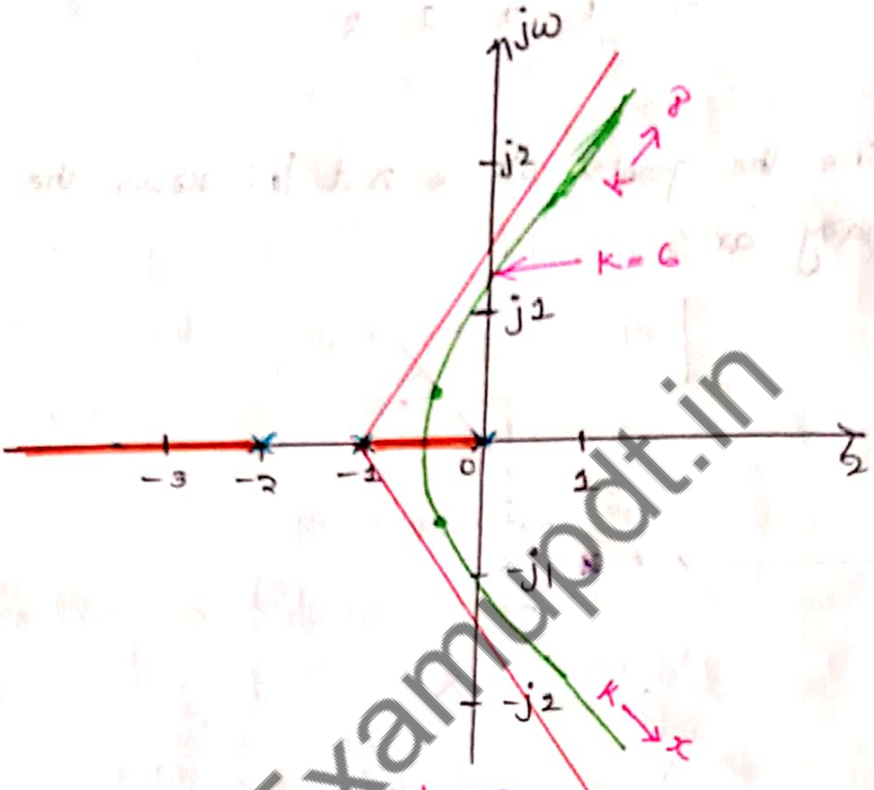
$$\Rightarrow (2\omega - \omega^3) \rightarrow \text{Imaginary part}$$

Equating both real and imaginary parts of this equation to zero.

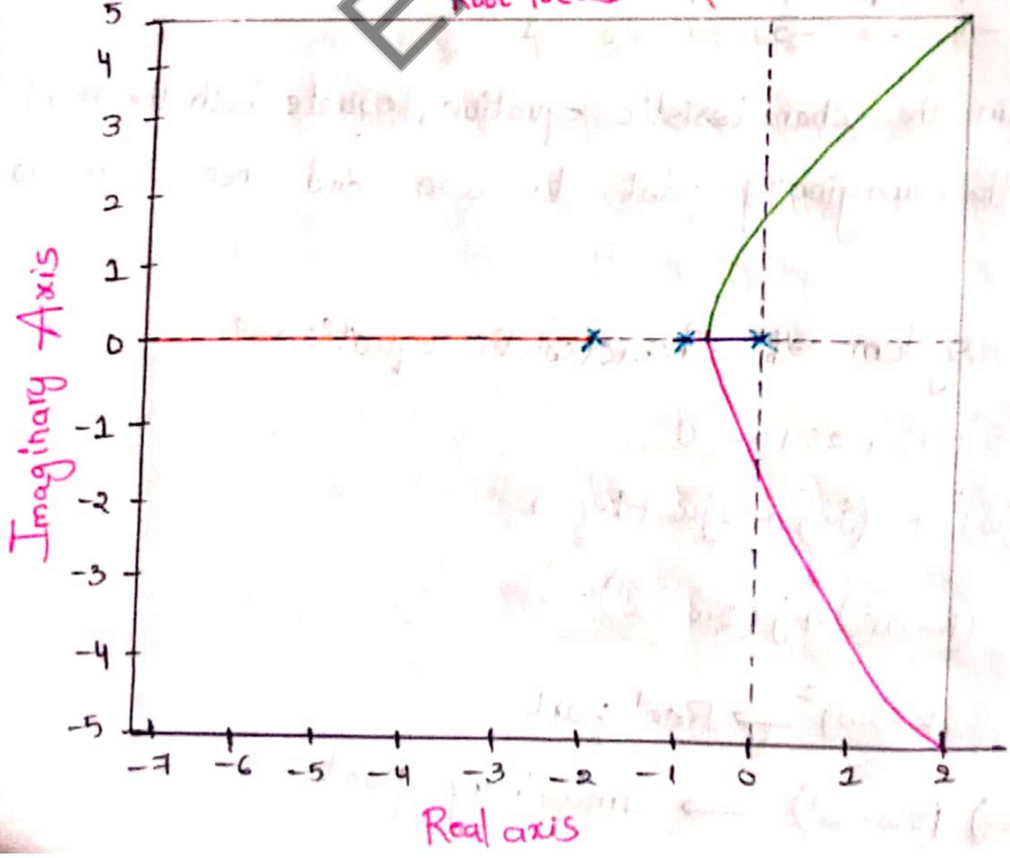
$$(2\omega-\omega^3)=0$$
$$(K-3\omega^2)=0$$

Which yields

$\omega=\pm\sqrt{2}$, $K=6$, (or) , $\omega=0$, $K=0$



Root locus



Example.

1) Determine the Breakaway and breakin points.

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

Sol:

$$1+G(s)H(s) = 0$$

$$G(s)H(s) = -1$$

$$\frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1$$

$$\frac{K(s^2-8s+15)}{(s^2+3s+2)} = -1$$

$$\boxed{K = \frac{-(s^2+3s+2)}{(s^2-8s+15)}}$$

Differentiation K with respect to s and setting the derivative equal to zero yields.

$$\frac{dK}{ds} = -\frac{[s^2-8s+15](2s+3) - [s^2+3s+2](2s-8)}{(s^2-8s+15)^2} = 0$$

$$11s^2-26s-61=0$$

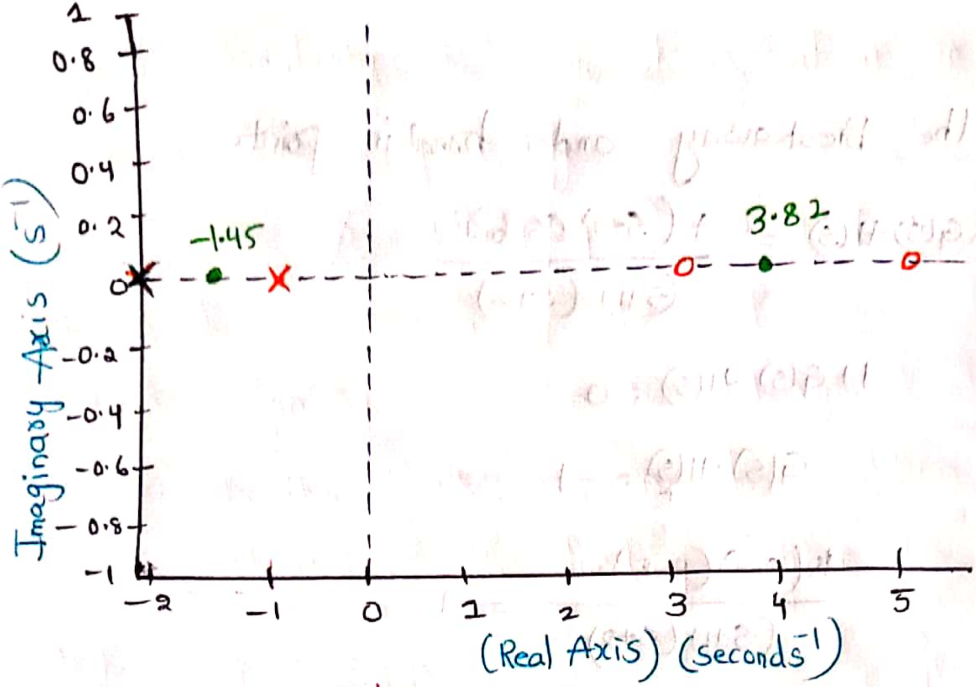
Hence, solving for s,

we find the breakaway and break-in points.

$$\boxed{s=-1.45 \quad \text{and} \quad 3.82}$$

$s=-1.45$, $\angle s-3 + \angle s-5 - \angle s+1 - \angle s+2 = -180-180-180-0 = -540$

$s=3.82$, $\angle s-3 + \angle s-5 - \angle s+1 - \angle s+2 = 0-180-0-0 = -180$



Pole-zero Map

- Breakaway point exists between 2 poles

i.e $\boxed{\text{breakaway} = -1.45}$

- Root locus moves towards the zeros.

$\boxed{\text{Breakin} = 3.82}$

Construction Rules:

Rule-1: The root locus is symmetrical about the real axis (σ axis).

* We know that the roots of the characteristic equation are either real (or) complex conjugate (or) combination of both. Therefore their locus must be symmetrical about the real axis of the s-plane.

Rule-2: As K increases from '0' to '∞' each branch of the root locus originates at an open loop pole with K=0 an terminates at either an open loop zero (or) on infinity with $K=\infty$. The number of branches terminating at infinity equals to number of open loop poles minus the zeros (n-m).

Rule-3: A point on the real axis lies on the locus if the number of open loop poles plus zeros on the real axis to the right of the point under consideration is odd.

Rule-4: The (n-m) branches of the root locus which tend to infinity, do so along straight line asymptotes whose angles are given by

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \; ; \; q = 0, 1, 2, \cdots (n-m-1)$$

Rule-5: The asymptotes cross the real axis at a point known as Centroid, determined by the relationship:

$$\frac{(\text{sum of real parts of poles} - \text{sum of real parts of zeros})}{\text{Number of poles} - \text{number of zeros}}$$

i.e.

$$\sigma = \frac{\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} Z_i}{n-m}.$$

Because all the Complex poles and zeros occur as a Conjugate pair, Centroid is always real quantity.

Rule-6: The breakaway points (or) breakin point (points at which multiple roots of a characteristic equation occur) of the root locus are the solutions of $\frac{dK}{ds} = 0$. The r branches of the root locus which meet at a point, breakaway at an angle of $\pm \frac{180^\circ}{r}$

Rule-7: The angle of departure of a root locus segment from a Complex open loop pole is given by

$$\phi_p = \pm 180^\circ(2q+1) + \phi \; ; \; q = 0, 1, 2$$

where ϕ is the net angle Contribution, at this pole of all other open loop zeros and poles.

Simalarly the angle of arrival at an open loop zero is given by

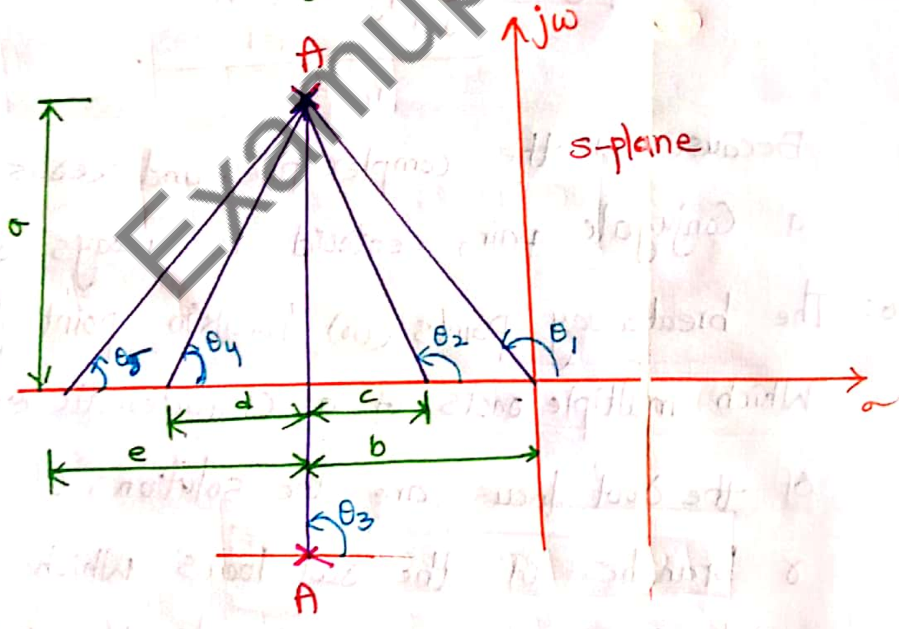
$$\phi_z = \pm 180^\circ(2q+1) - \phi \;;\; q = 0,1,2.$$

Rule-8: The intersection of the root locus with the imaginary axis can be determined by the use of the Routh criterion. The condition for 'K' thus obtained is the limiting condition for stability.

Angle of Departure (from a complex pole A) $= 180^\circ -$ [Sum of angles of vectors to the complex pole A from other poles] $+$ [Sum of angles of vectors to the complex pole A from zeroes]

Note:

The angles can be calculated as shown in figure (or) they can be measured ~~during~~ using protractor.



$$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{a}{b}\right)$$

$$\theta_2 = 180^\circ - \tan^{-1}\left(\frac{a}{c}\right)$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\left(\frac{a}{d}\right)$$

$$\theta_5 = \tan^{-1}\left(\frac{a}{e}\right)$$

Problem:

The transfer function of feedback loop control system is given by $G(s)H(s) = \frac{K}{s(s+4)(s+6)}$

Sol:

Step-1: When $K=0$, the point on the root loci are at poles of $G(s)H(s)$. Poles are at $0, -4$ and -6.

Step-2: When $K=\infty$, the points if the root locus are at the zeros of $G(s)H(s)$. Since there are no open loop zeros, the root loci terminate at infinite zeros.

Step-3: The number of root loci is $N-n$, where n is no of poles i.e 3

Step-4: The root loci are symmetrical about the real axis. The root locus is present on the real axis whenever the total number of poles and zeros to the right of the section is odd.

- Root loci are present between the poles 0 and -4. (Any section between these poles have 1 no. of poles to the right)
- Root loci is not present between -4 and -6. To the right of this section, the total no. of poles and zeros is even
- Root locus is present between -6 and $-\infty$ as the total number of poles and zeros is off to the right of any point in the segment.

Step-5: Determine the angle of asymptotes $\theta_K = \frac{(2q+1)180^\circ}{n-m}$

- Here q varies from 0 to 2

$$\theta_1 = \frac{(2\times 0+1)180^\circ}{3} = 60^\circ$$

$$\boxed{\theta_1 = 60^\circ}$$
$$\boxed{\theta_2 = 180^\circ}$$
$$\boxed{\theta_3 = 300^\circ}$$

Step-6: Centroid $\sigma = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{n-m} = \frac{(0-4-6)-0}{3}$

$$\boxed{\sigma = -3.33}$$

Step-7: To find the break away point. We can find the breakaway point from the characteristic equation $1+G(s)H(s)=0$

$$1+\frac{K}{s(s+4)(s+6)} = 0$$

$$s(s+4)(s+6)+K=0$$

$$s^3+10s^2+24s+K=0$$

$$\boxed{K = -(s^3+10s^2+24s)}$$

$$\frac{dK}{ds} = -[3s^2+20s+24] = 0$$

$$\boxed{\frac{dK}{ds} = 3s^2+20s+24=0}$$

∴ The roots are

$$s_{1,2} = \frac{-20 \pm \sqrt{(20)^2-4(3)(24)}}{2(3)}$$

$$\boxed{s_{1,2} = -1.56 \text{ and } -5.09}$$

- Select the values of s where real axis loci in step-4 are present. Real axis loci is present between 0 and −4. So we select $\beta = -1.56$

Breakaway point is −1.56.

Step-8: Angle of departure (or) arrival is not applicable as there are no Complex Conjugate poles (or) zeros respectively

Step-9: The $j\omega$ axis crossing, to find the Value of K for stability and a interaction with imaginary axis, We have the characteristic equation.

$$1 + G(s)H(s) = 0$$

$$s^3 + 10s^2 + 24s + K = 0$$

On Imaginary axis $s = j\omega$

$$-j\omega^3 - 10\omega^2 + j24\omega + K = 0$$

$$(K - 10\omega^2) + j\omega(24 - \omega^2) = 0$$

Imaginary part $= 0$

$$\omega(24 - \omega^2) = 0$$

$$\boxed{\omega = 0}$$

$$24 - \omega^2 = 0$$

$$\omega^2 = 24$$

$$\omega = \sqrt{24}$$

$$\boxed{\omega = 4.9}$$

Real part $= 0$

$$K - 10\omega^2 = 0$$

$$K - 10(24) = 0$$

$$\boxed{K = 240}$$

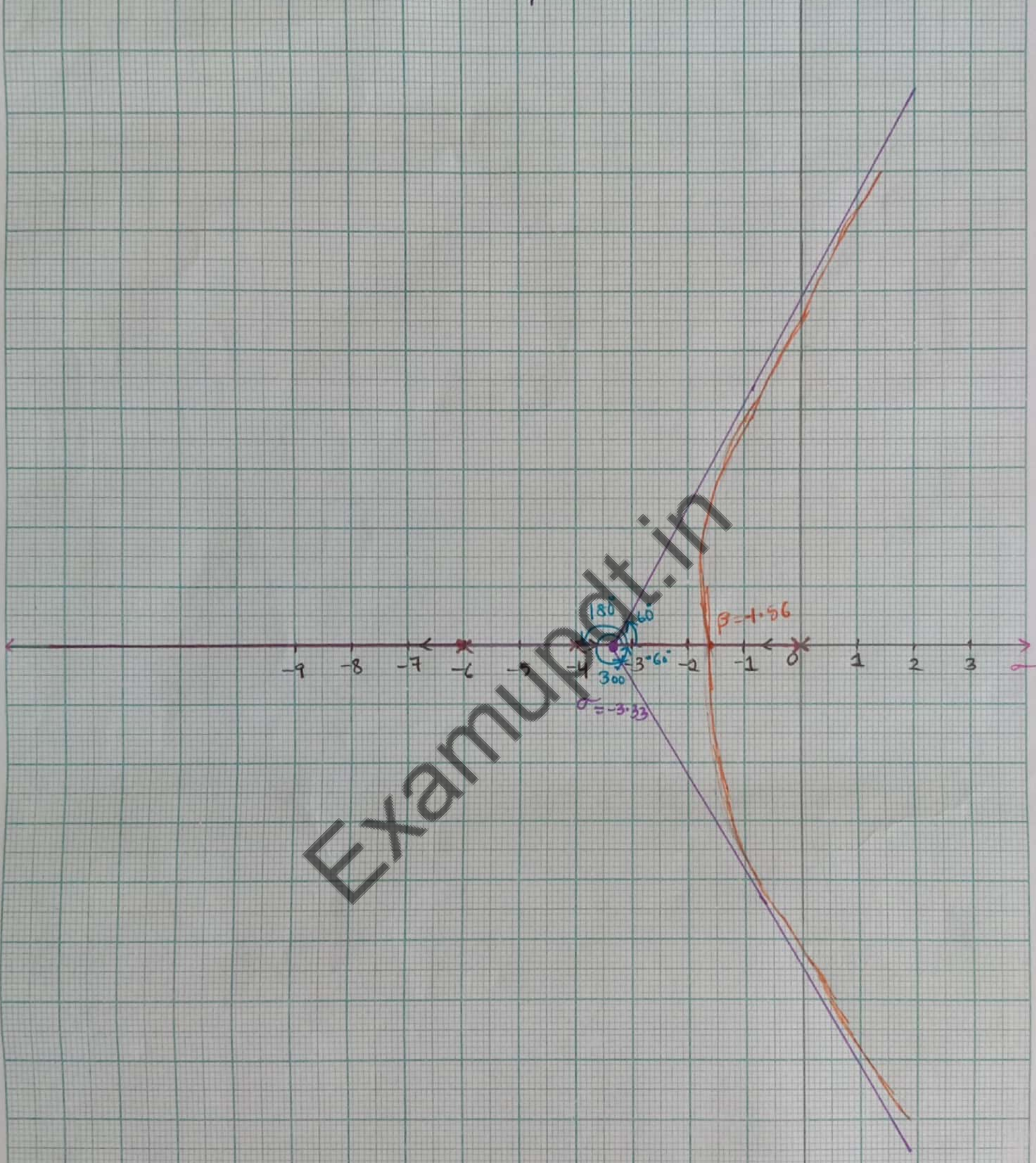
$K < 240 \longrightarrow$ System is stable

$K = 240 \longrightarrow$ Marginally state, oscillations of frequency $\omega = 4.9$ rad/s

$K > 240 \longrightarrow$ system is unstable.

1

scale:
on x-axis 1 unit = 1 centroid point.



$$G(s)H(s) = \frac{K}{s(s+4)(s+6)}$$

$\beta = -1.56$

$\sigma = -3.33$

Problem:

Sketch the root locus for the transfer function given by

$$G(s)H(s) = \frac{k(s+2)}{s(s+4)(s+6)}$$

Step-1: When $k=0$, the points on the root loci are at the poles of $G(s)H(s)$. The poles are $0, -4, -6$.

Step-2: When $k=\infty$, the points of the root loci are at the zeros of $G(s)H(s)$. The root loci starts from the poles and ends at zeros $-2, \infty, \infty$

Step-3: The number of root loci is $N=n=3$

Step-4: The root loci are symmetrical about the real axis and is present on the real axis whenever, the no. of poles and zeros to the right side of a section is odd.

- Root locus is present between 0 and -2.
- Root locus is present between -4 and -6.
- Root locus is not present between -2 & -4 and also from -6 to $-\infty$

Step-5: Determine the angle of the asymptotes

$$\theta_K = \frac{(2q+1)180^\circ}{n-m} = 90^\circ \text{ and } 270^\circ$$

Step-6: Centroid

$$\sigma = \frac{\Sigma poles - \Sigma zeros}{n-m} = \frac{(0-4-6)-(-2)}{3-1}$$

$$\boxed{\sigma = -4}$$

Step-7: To find the breakaway point, we have

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k(s+2)}{s(s+4)(s+6)} = 0$$

$$s(s+4)(s+6) + K(s+2) = 0$$

$$K(s+2) = -(s^3+10s^2+24s)$$

$$\boxed{K = \frac{-(s^3+10s^2+24s)}{\cancel{K}(s+2)}}$$

Differentiating the above equation.

$$\frac{dK}{ds} = \frac{-\left[(s+2)(3s^2+20s+24) - (s^3+10s^2+24s)\right]}{(s+2)^2} = 0$$

$$\frac{dK}{ds} = \frac{-3s^3-20s^2-24s-6s^2-40s-48+s^3+10s^2+24s}{(s+2)^2} = 0$$

$$\frac{dK}{ds} = \frac{-2s^3-16s^2-40s-48}{(s+2)^2} = 0$$

$$\frac{dK}{ds} = 2s^3+16s^2+40s+48 = 0$$

$$\boxed{\frac{dK}{ds} = s^3+8s^2+20s+24 = 0}$$

On solving we get

$$\boxed{\beta = -4.9}$$

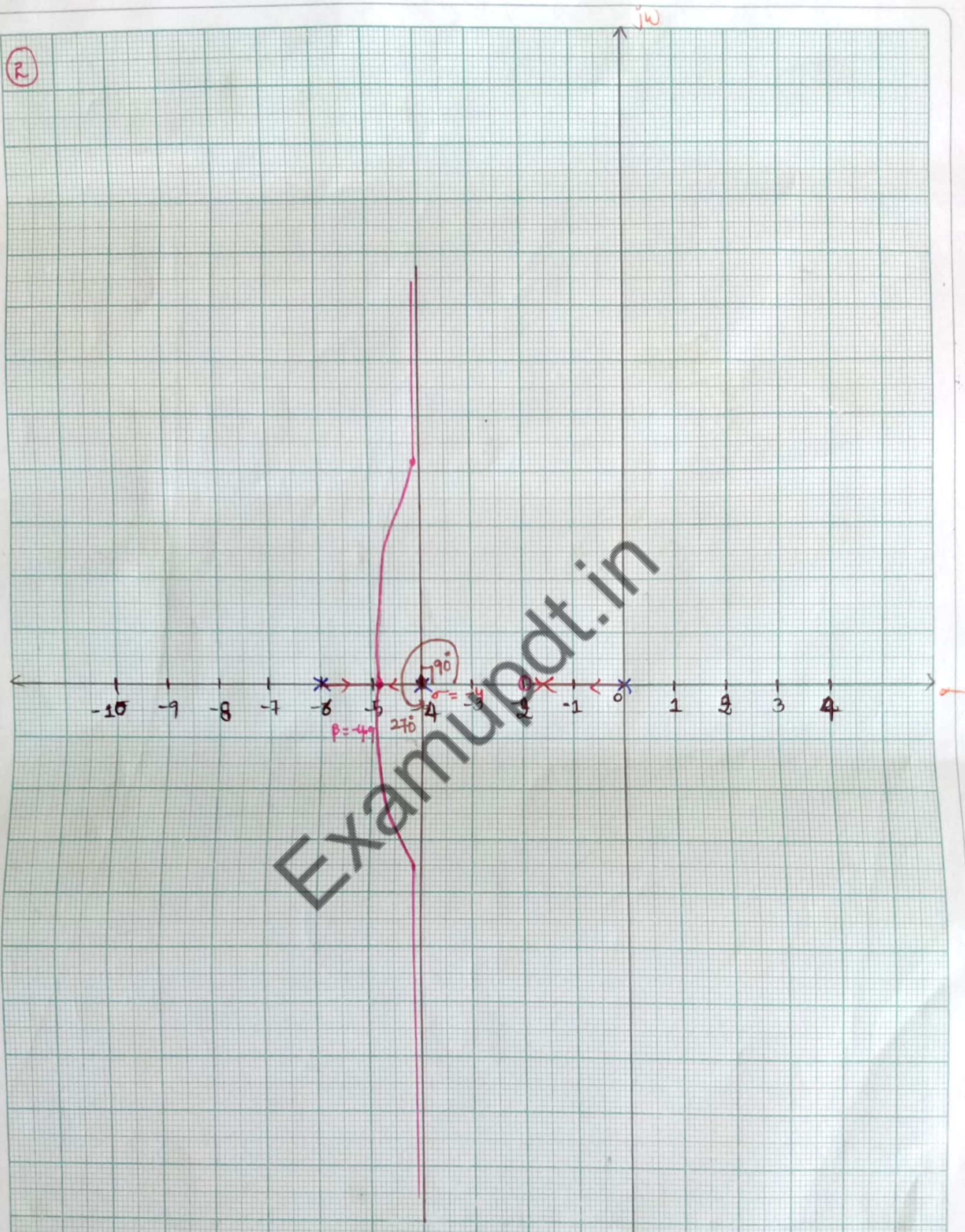
Remaining neglected due to nd presence of root locus.

Step-8: Angle of depature is not applicable as the poles and zeros are real numbers.

step9: The jω axis crossing is obtained using the Routh array.

By adding a zero to open-loop System. We made the root locus to move left and make the system stable.

2



$$G(s)H(s) = \frac{K(s+2)}{s(s+4)(s+6)}$$

$\beta = -4.9$ $\quad \sigma = -4$

3) Sketch the root locus for the transfer function

$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)(s+4)}$$

Sol:-

$n=4, \quad m=0$

Step-1: When $K=0$ the points on the root loci are at the poles of $G(s)H(s)$. The poles are $-1, -2, -3, -4$.

Step-2: When $K=\infty$, the points of the root loci are at the zeros of $G(s)H(s)$. The root loci starts from the poles and ends at zeros $\infty, \infty, \infty, \infty$.

Step-3: The number of root loci is $N=n=4$

Step-4: The root loci are symmetrical about the real axis and is present on the real axis whenever, the no. of poles and zeros to the right side of a section is odd.

- Root locus is present between -1 and -2
- Root locus is present between -3 and -4.
- Root loci are not present between -2 and -3 & also from -4 to $-\infty$.

Step-5: Determine the angle of the asymptotes

$$\theta_K = \frac{(2q+1)180^\circ}{n-m} = 45^\circ, 135^\circ, 225^\circ \text{ and } 315^\circ$$

Step-6: Centroid

$$\sigma = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{n-m} = \frac{(-1-2-3-4)-(-0)}{4-0}$$

$$\boxed{\sigma = -2.5}$$

Step-7: To find the breakaway point, we have $1+G(s)H(s)=0$

$$1+G(s)H(s)=0$$

$$1+\frac{K}{(s+1)(s+2)(s+3)(s+4)}=0$$

$$(s+1)(s+2)(s+3)(s+4)+K=0$$

$s^4 + 10s^3 + 35s^2 + 50s + 24 + K = 0$.

$$\boxed{K = -\left[s^4 + 10s^3 + 35s^2 + 50s + 24\right]}$$

Differentiating the above equation

$$\frac{dK}{ds} = -\left[4s^3 + 30s^2 + 70s + 50\right] = 0$$

$$\boxed{\frac{dK}{ds} = -\left[2s^3 + 15s^2 + 35s + 25\right] = 0}$$

Solving by synthetic division one of the root is -1.38 and another root is at -3.618 are the two breakaway points.

Step-8: Angle of departure is not applicable as the poles and zeros are real numbers.

Step-9: The $j\omega$ axis crossing is obtained using the Routh Array

$s^4 + 10s^3 + 35s^2 + 50s + (24+K) = 0$

s^4	1	35	24+K
s^3	10	50	
s^2	30	24+K	
s^1	$\frac{126-K}{3}$		
s^0	24+K		

$$\begin{vmatrix} 1 & 35 \\ 10 & 50 \end{vmatrix} = \frac{(10\times 35) - (1\times 50)}{10}$$

RH criterion

Elements of first column must be of same sign.

$$\frac{126-K}{3} > 0$$

$$126 - K > 0$$

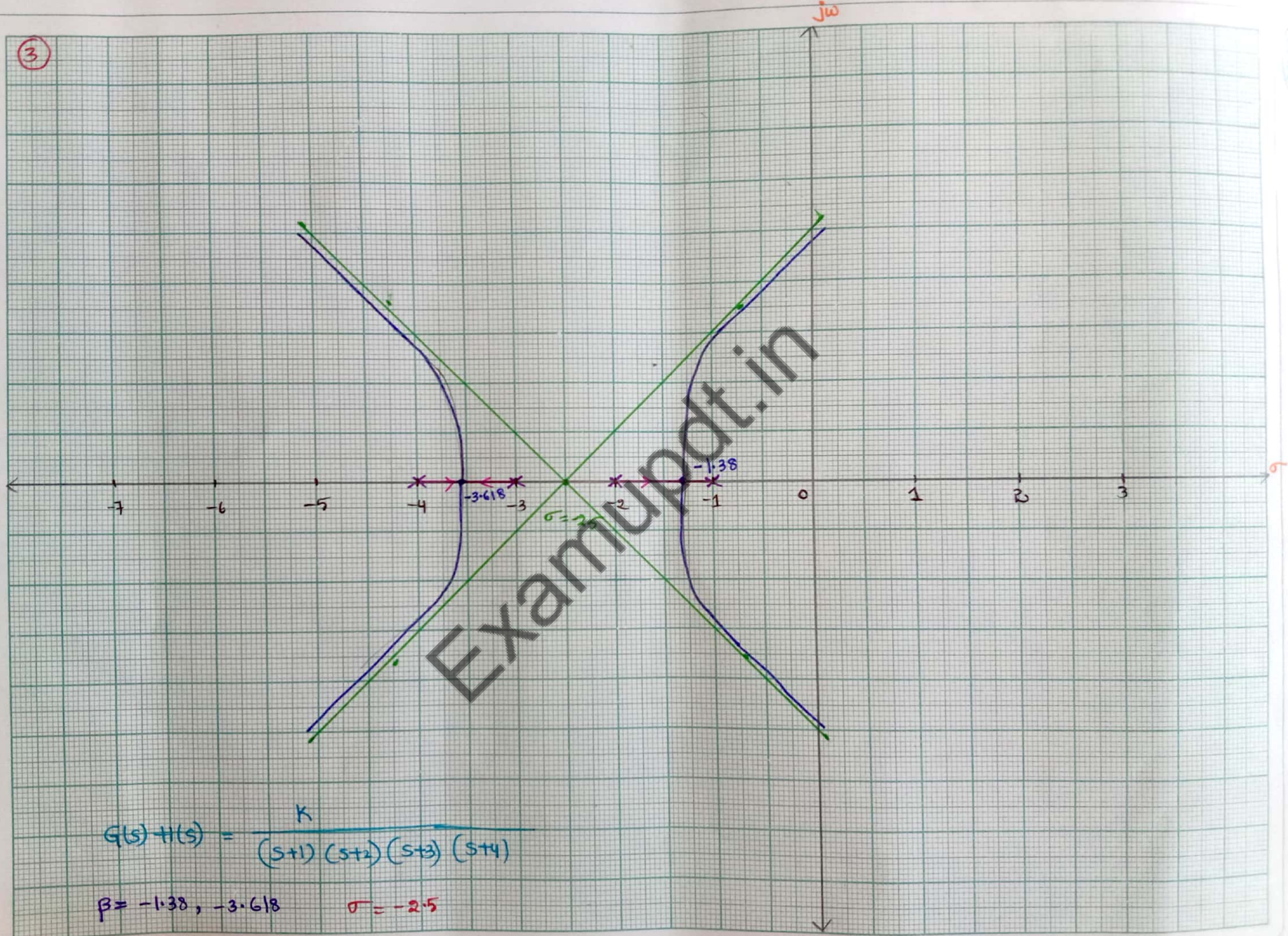
$$\boxed{K < 126}$$

$$24 + K > 0$$

$$\boxed{K > -24}$$

$$\boxed{-24 < K < 126}$$

③



$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)(s+4)}$$

$\beta = -1.38, -3.618$ $\sigma = -2.5$

put $K=126$

Auxiliary equation $A[s] = 30s^2 + 24 + K = 0$

$$A[s] = 30s^2 + 126 + 24 = 0$$

$$A[s] = 30s^2 + 150 = 0$$

$$A[s] = 30(s^2+5) = 0$$

$$30(s^2+5) = 0$$

$$s^2 = -5$$

$$\boxed{s = \pm j\sqrt{5}}$$

Problem:

Construct a root locus plot for the transfer function

$$G(s)H(s) = \frac{K}{s(s^2+2s+2)}$$

Sol:

The denominator consists of quadratic equation, factoring the equation we get the roots as

$$s_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$$

Hence the equation represents the complex conjugate poles and can be written as

$$G(s)H(s) = \frac{K}{s(s+1-j)(s+1+j)}$$

Step-1: When $K=0$, the point on the root loci are at the poles of $G(s)H(s)$. The poles are $0, -1+j, -1-j$

Step-2: When $K=\infty$, the points on the root loci terminate at the zeros of $G(s)H(s)$ at ∞, ∞, ∞.

Step-3: The number of branches of root loci

$$N = n = 3.$$

Step-4: The root loci is symmetrical about the real axis. The loci is present on the real axis. Whenever the total no. of poles and zeros to the right of the Section is odd. The root locus is present from 0 to $-\infty$ on the negative real axis.

Step-5: Determine the angle of the asymptotes.

$$\theta_K = \frac{(2q+1)\,180^\circ}{n-m} = 60^\circ,\ 180^\circ \text{ and } 300^\circ$$

Step-6: Centroid

$$\sigma = \frac{\Sigma(\text{Poles}) - \Sigma(\text{zeros})}{n-m} = \frac{(0-1-j-1+j)-(0)}{3-0}$$

$$\boxed{\sigma = -0.66}$$

Step-7: To find the break away point, We have

$$1+G(s)H(s)=0$$

$$1+\frac{K}{s(s^2+2s+2)}=0$$

$$s^3+2s^2+2s+K=0$$

$$\therefore K = -\left[3s^3+s^2+2s+k\right]$$

$$\boxed{K = -\left[s^3+2s^2+2s+k\right]}$$

Differerentiating the above equation.

$$\frac{dk}{ds} = -\left[3s^2+4s+2\right]=0$$

$$\boxed{\frac{dk}{ds} = 3s^2+4s+2=0}$$

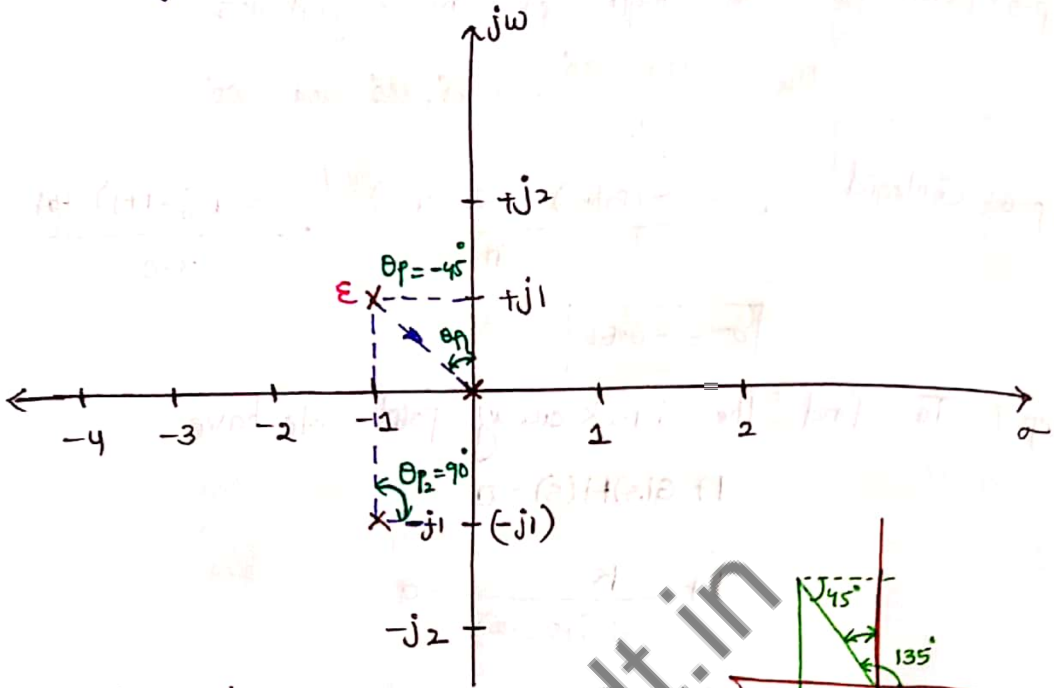
Roots are imaginary and there is no breakaway points on the real axis

Step-8: Angle of departure is applicable as the poles are Complex Conjugate.

- Consider the pole at $-1+j$ and connect lines from all the open loop zeros to S_1. Let the measured angles be

$\theta_{p_1}, \theta_{p_2}, \theta_3$ etc in the anti clockwise direction

Step-9: The jω axis crossing is determined by the R-H criterion using characteristic equation.



θ_a = 180° − (angle with zeros) + (angle with poles).

θ_d = 180° − (angle made by point with all other poles) + (angle made by point with all other zero).

$\theta_d = 180° - (135 + 90)$

$\boxed{\theta_d = -45°}$

$s^3 + 2s^2 + 2s + K = 0$

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	
s^0	K	

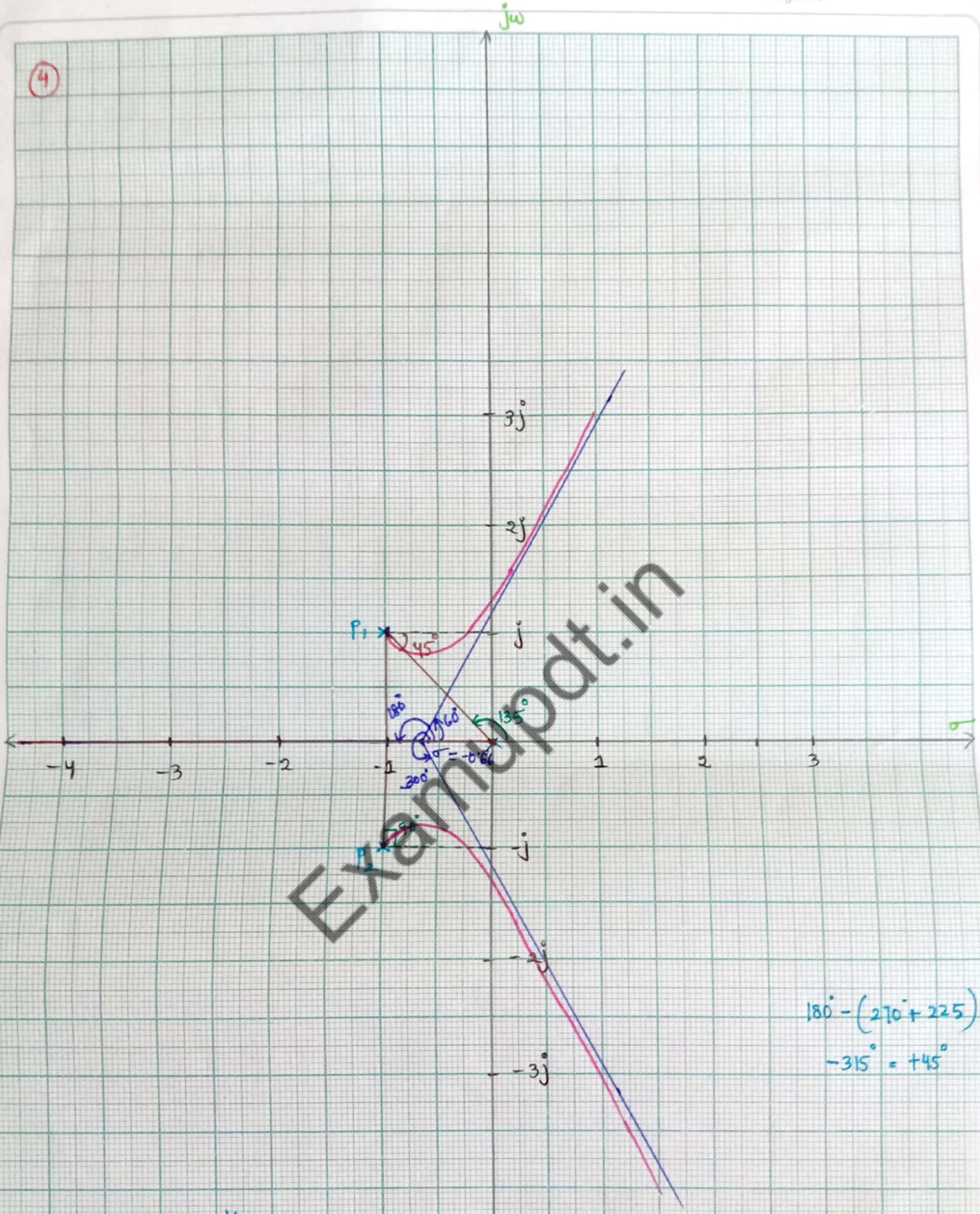
$\frac{4-K}{2} > 0$

$\therefore \boxed{K > 0}$

$4 - K > 0$

$\therefore \boxed{K < 4}$ $\quad \therefore \boxed{0 < K < 4}$

(4)



$$G(s)H(s) = \frac{K}{s(s^2+2s+2)}$$

$$\sigma = -0.66$$

$$s^3+2s^2+2s+4=0$$

$$s^2(s+2)+2(s+2)=0$$

$$(s+2)(s^2+2)=0$$

When $K=4$ it crosses imaginary axis.

$$s^2+2=0$$

$$\boxed{s=\pm j\sqrt{2}}$$

Problem:

Sketch the root locus of the control system having $G(s)H(s) = \frac{K(s^2-2s+5)}{(s+2)(s-0.5)}$. Find the maximum and minimum values of K for stability and the value of K that gives the system characteristic equation of damping ratio of 0.5.

Sol: **Step-1:** When $K=0$, the point on the root loci are at the poles of $G(s)H(s)$ i.e $-2, 0.5$.

Step-2: When $K=\infty$, the points on the root loci terminate at the zeros of $G(s)H(s)$ at $1-2j$ and $1+2j$.

Step-3: The number of branches of root loci $N=n$

$$N=n=2$$

Step-4: The root loci is symmetrical about the real axis. The loci is present on the real axis when ever the total no. of poles and zeros to the right of the section is odd. The root locus is present from 0.5 to -2 on the real axis. The root locus is not present elsewhere on the real axis.

Step-5: Determine the angle of the asymptotes

$$\theta_K = \frac{(2q+1)180^\circ}{n-m}$$

As $n-m$, no asymptotes exist.

Step-6: Centroid $\sigma = \frac{\Sigma(\text{Poles}) - \Sigma(\text{zeros})}{n-m}$. As there are no asymptotes, no centroid exists.

Step-7: To find the break away point, We have

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s^2-2s+5)}{(s+2)(s-0.5)} = 0$$

$$(s+2)(s-0.5) + K(s^2-2s+5) = 0$$

$$s^2+1.5s-1 + K(s^2-2s+5) = 0$$

$$\boxed{K = -\frac{(s^2+1.5s-1)}{s^2-2s+5}}$$

Differentiating the above equation.

$$\frac{dK}{ds} = -\frac{\left[(s^2-2s+5)(2s+1.5) - (s^2+1.5s-1)(2s-2)\right]}{(s^2-2s+5)^2} = 0$$

$$\frac{dK}{ds} = -\frac{(s^2-2s+5)(2s+1.5) + (s^2+1.5s-1)(2s-2)}{(s^2-2s+5)^2} = 0$$

$$\boxed{\frac{dK}{ds} = 3.5s^2 - 12s - 5.5 = 0}$$

- There are two breakaway points

$$s_{1,2} = \frac{12 \pm \sqrt{(12)^2 - 4\times3.5(-5.5)}}{2\times3.5}$$

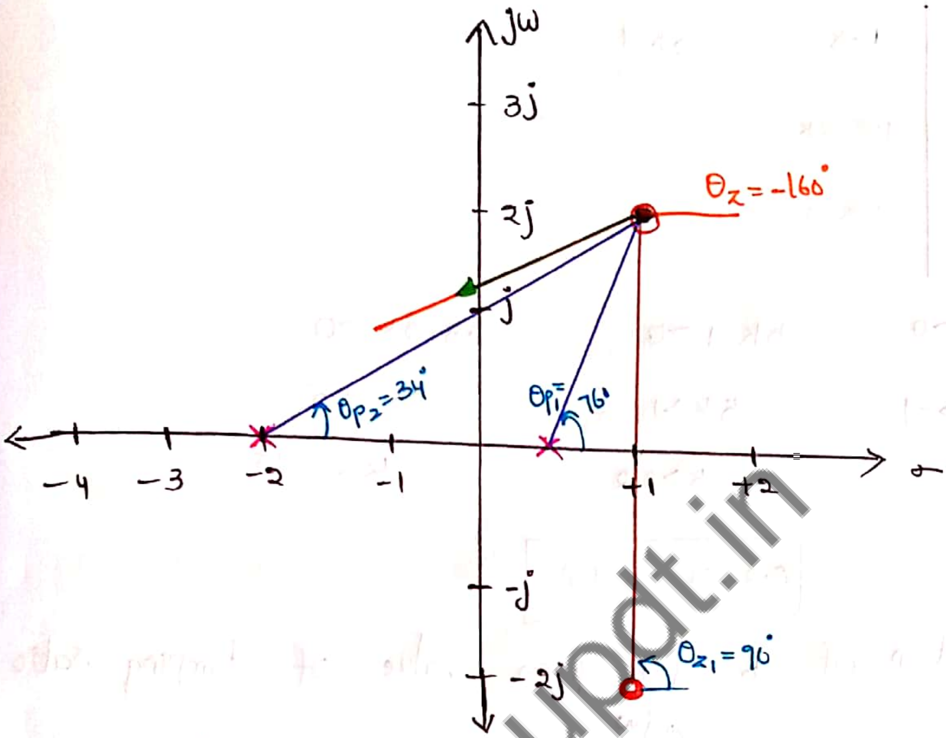
$$\boxed{s_{1,2} = -0.409,\ 3.86}$$

- The Valid breakaway point is

$$\boxed{\beta = -0.409}$$

Step-8: Angle of arrival is applicable as there are two complex Conjugate zeros.

- Consider the zero at $1+2j$ and Connect lines from all the open loop and zeros to S_1. Let the measured angles be $\theta_{P_1}, \theta_{P_2}, \theta_{z_1}$. etc in the anticlockwise direction.



From figure,

$$\theta_{P_1} = \tan^{-1}\left(\frac{2}{0.5}\right) = 76^\circ$$

$$\theta_{P_2} = \tan^{-1}\left(\frac{2}{3}\right) = 34^\circ$$

$$\theta_{z_1} = 90^\circ$$

Using formula for angle of arrival

$$\theta_z = -180^\circ - (\text{angle with zeros}) + (\text{angle of poles})$$

$$\theta_z = -180^\circ - 90 + 76 + 34$$

$$\theta_z = -270 + 110$$

$$\therefore \boxed{\theta_z = -160^\circ}$$

Step-9: The jw axis crossing is obtained from RH-criterion.

Characteristic equation,

$$(s+2)(s-0.5)+K(s^2-2s+5)=0$$

$$(s^2+1.5s-1)+K(s^2-2s+5)=0$$

$$(1+K)s^2+(1.5-2K)s+(5K-1)=0$$

s^2	$1+K$	$5K-1$
s^1	$1.5-2K$	
s^0	$5K-1$	

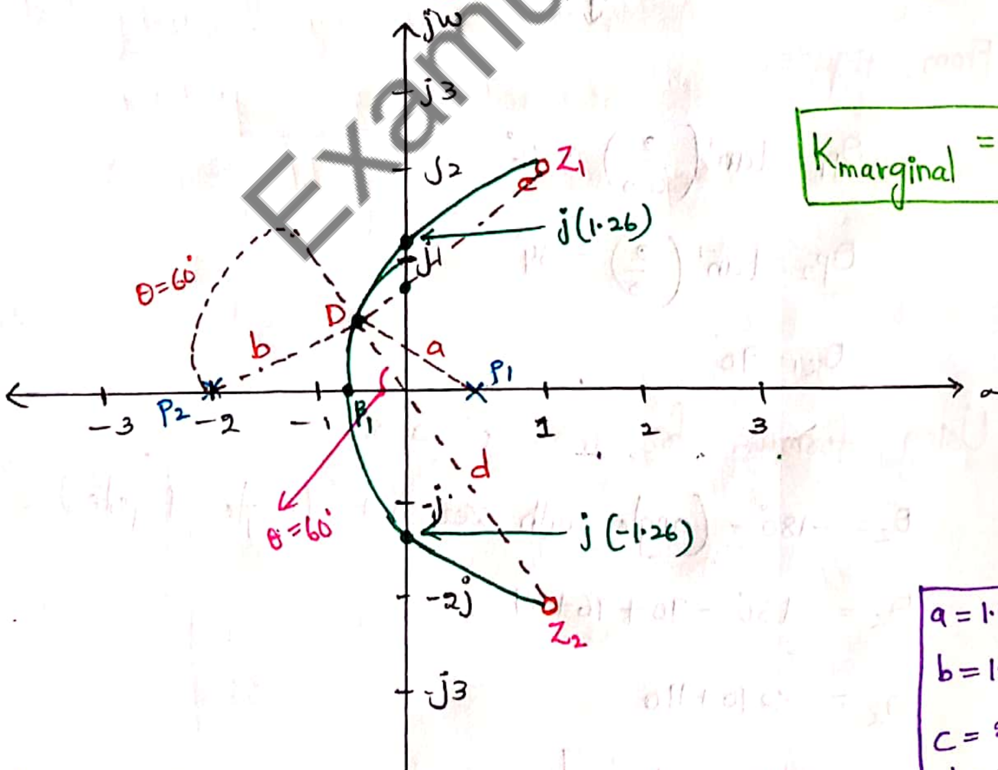
$1+K>0$ $\quad$ $5K-1>0$ $\quad$ $1.5-2K>0$

$K>-1$ $\quad$ $5K>1$ $\quad$ $2K<1.5$

$K>0.2$ $\quad$ $K<0.75$

$\therefore \boxed{0.2<K<0.75}$

- Determination of K for given value of damping ratio



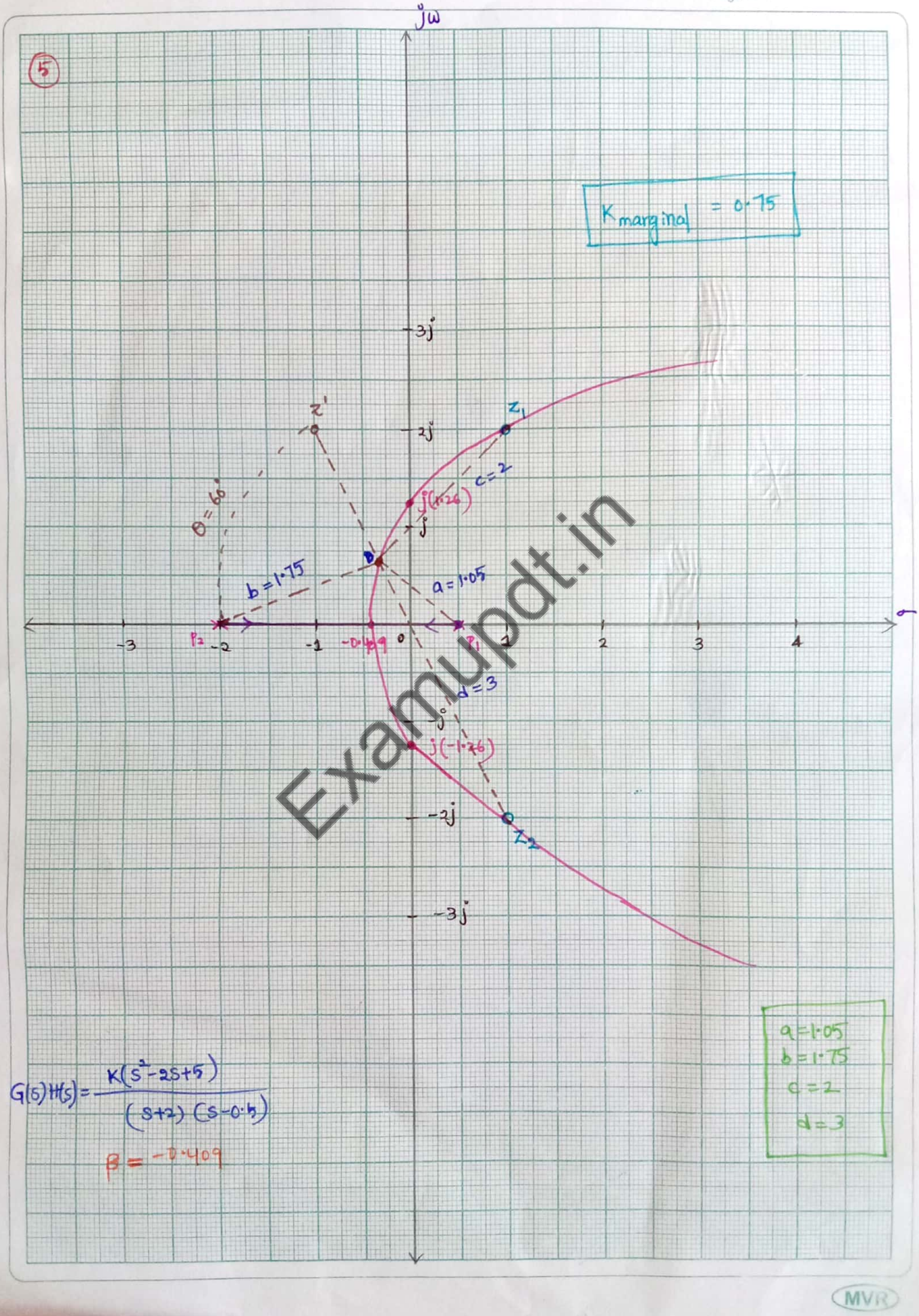
$\boxed{K_{marginal}=0.75}$

$a=1.05$
$b=1.75$
$c=2$
$d=3$

- Let $\theta=\cos^{-1}\xi=\cos^{-1}(0.5)=60°$

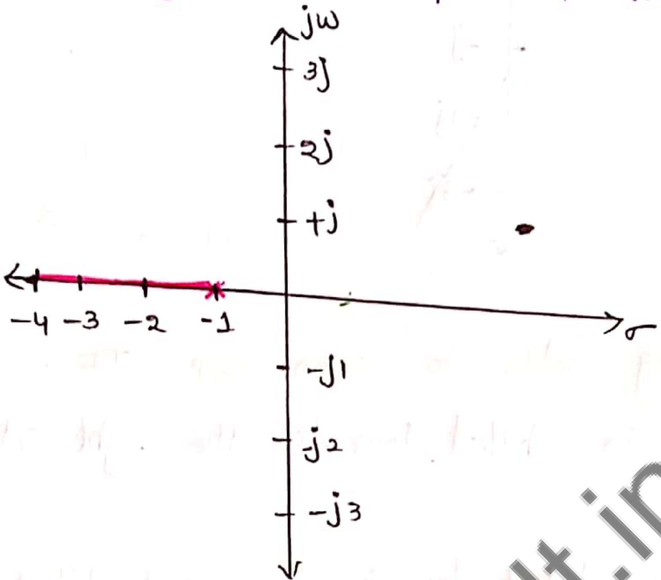
Draw a line at angle of 60° with respect to real axis about the origin. This line intersect the root locus at a point D

Page No.

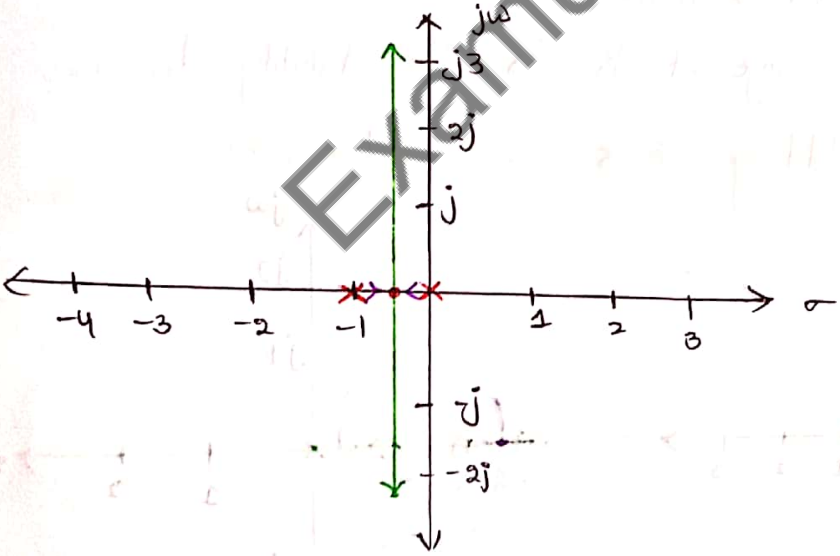

5
jω
σ
$K_{marginal} = 0.75$
3j
2j
j
-j
-2j
-3j
-3
-2
-1
0
1
2
3
4
P_2
P_1
-0.409
Z'
Z_1
Z_2
θ = 60°
b = 1.75
a = 1.05
c = 2
d = 3
j(1.76)
j(-1.76)
$G(s)H(s) = \frac{K(s^2-2s+5)}{(s+2)(s-0.5)}$
β = -0.409
a = 1.05
b = 1.75
c = 2
d = 3

- Find the Coordinates of D and Substitute in the characteristic equation to determine the Value of k at the point D.
- therefore, for $K = 0.3$ the system has a damping ratio of 0.5.

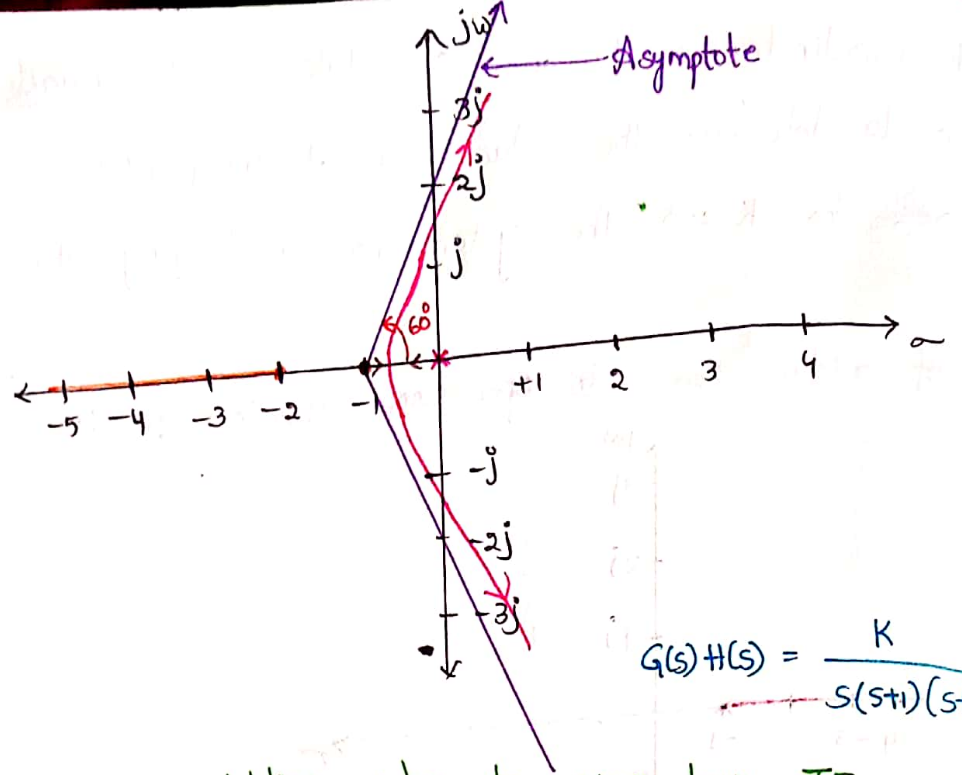
Effects of adding poles to open Loop Transfer function.



$$G(s)H(s) = \frac{K}{S+1}$$



$$G(s)H(s) = \frac{K}{s(s+1)}$$

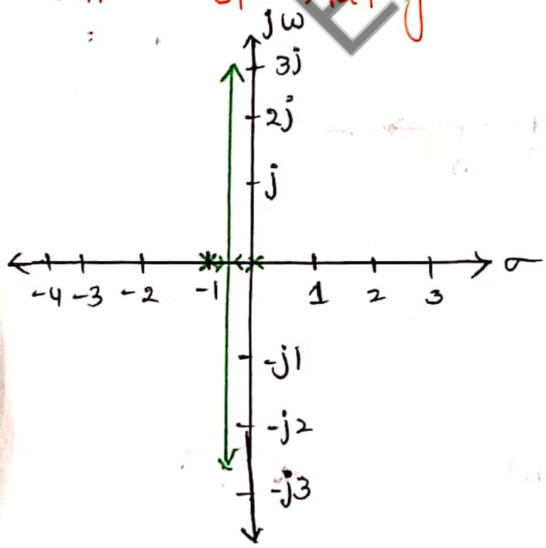


$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

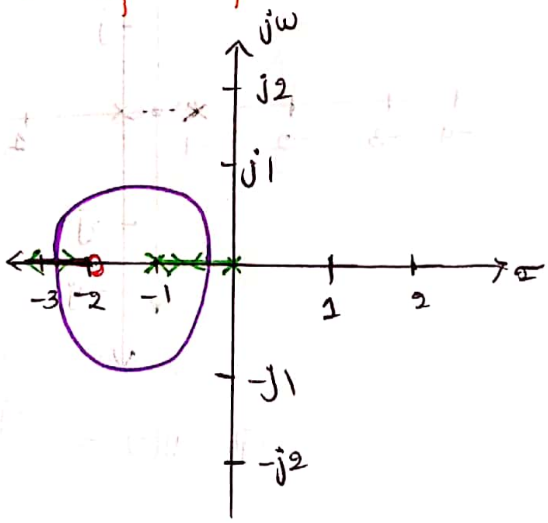
Effects of Adding poles to Open Loop TF

- The root locus is shifted towards the right side of the s-plane
- As root locus is shifted to the imaginary axis, the system becomes more oscillatory in nature
- The system becomes more unstable.
- The operating range of K for the stability decreases.

Effects of Adding Zeros to Open Loop T.F



$$G(s)H(s) = \frac{K}{s(s+1)}$$



$$G(s)H(s) = \frac{K(s+2)}{s(s+1)}$$

- The root locus is shifted towards the left side of the s-plane
- As root locus moves away from the imaginary axis and the system becomes less oscillatory in nature.
- The system becomes relatively more stable.
- The operating range of K for the stability increases.

Advantages of Root locus

- The operating range of the system gain K can be obtained for designing a system with absolute stability.
- The roots can be mapped in s-plane so as to determine the absolute stability of the system.
- The limiting value of K for marginal stability can be determined using root locus technique.
- It is possible to choose the gain according to the desired damping ratio and vice versa.
- Gain margin and phase margin of the system can be determined.
- Various time parameters, such as rise time, settling time etc can be determined from the root locus.