Unit-3: Frequency - Response Analysis

Introduction

- Frequency respose is the steady-state response of a system to a sinusoidal input
- In Frequency-response methods, the frequency of the input signal is varied over a certain range of the resulting response is studied.

Concept of Frequency Response

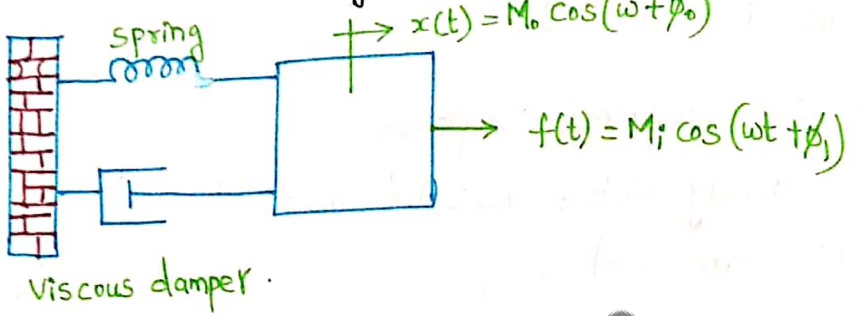
- In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency.
- Even though these responses are of the same frequency as the input, they differ in amplitude and phase angle from the input.
- These differences are functions of frequency.

Why frequency Response?

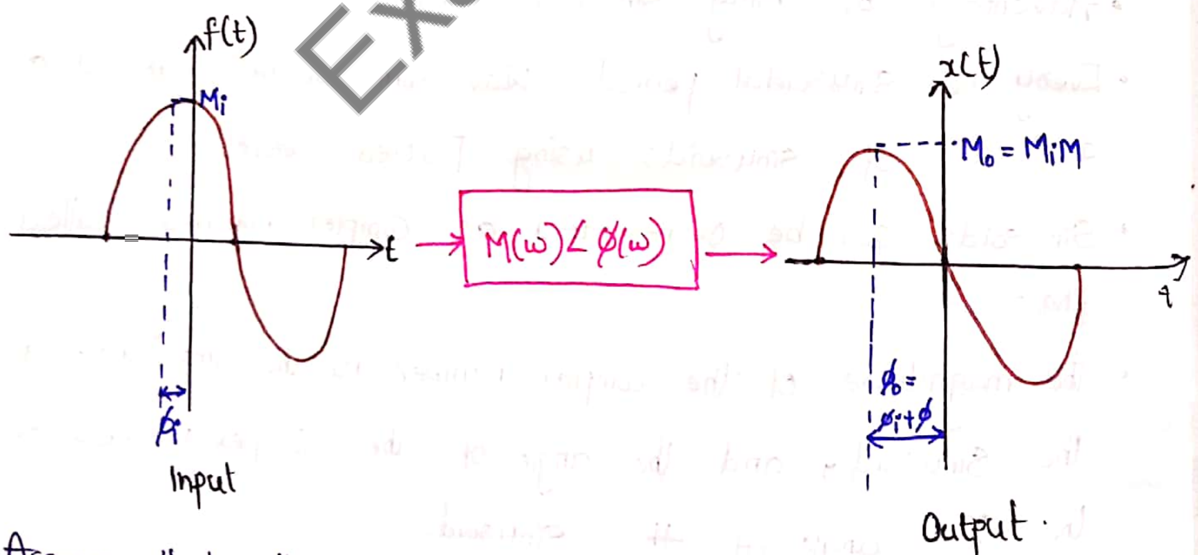
- Advantage of using sinusoidal functions.
- Every non-sinusoidal periodic waveform can be expressed as a series of sinusoids using Fourier Series.
- Sinusoids can be represented as complex numbers called phasors.
- The magnitude of the complex number is the amplitude of the sinusoid, and the angle of the complex number is the phase angle of the sinusoid.
- Thus $M_1 \cos(\omega t + \phi)$ can be represented as $M_1 \angle \phi$, where the frequency (ω) is implicit.

- A system causes both the amplitude and phase angle of the input to be changed.
- Therefore, the system itself can be represented by a complex number.
- Thus, the product of the input phasor and the system function yields the phasor representation of the output.

Consider the mechanical system,



If the input force $f(t)$ is sinusoidal, the steady-state output response $x(t)$ of the system is also sinusoidal and at the same frequency as the input.



- Assume that the system is represented by the complex number $M(\omega)\angle\phi(\omega)$.
- The output is found by multiplying the complex number representation of the input by the complex number representation

of the system.

- Thus, the steady-state output sinusoid is

$$\boxed{M_o(\omega)\angle\phi_o(\omega) = M(\omega)M_i(\omega)\angle[\phi(\omega)+\phi_i(\omega)]}$$

Where $M_o(\omega)$ is the magnitude response

$\phi_o(\omega)$ is the phase response.

The combination of the magnitude and phase frequency responses is called the frequency response.

Sinusoidal Input and Response to Sinusoidal Input:

- Consider a linear system with sinusoidal input $r(t) = A\sin\omega t$
- Under steady state, the system output as well as the signals at all other points in the system are sinusoidal. The output may be written as

$$\boxed{c(t) = B\sin(\omega t+\phi)}$$

- The magnitude and phase relationship between the sinusoidal input and the steady state output is termed the frequency response.
- In LTI systems, the frequency response is independent of the amplitude and the phase of the input signal.
- Frequency response is the test on a system performed by keeping the amplitude of input fixed and determining the amplitude and phase of the output for a suitable range of frequencies.
- Signal generators and precise measuring instruments are readily available for various ranges of frequencies and amplitudes

• Ease and accuracy of measurement is an advantage in frequency response.

Relationship between Time and Frequency response

What is Frequency Response?

∴ The response of a system can be divided into transient response and the steady state response. The steady state response of a system for an input sinusoidal signal is known as the Frequency Response.

• If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

Relationship between Time Response and Frequency Response:

Consider an input sinusoidal signal

$$r(t) = A\sin(\omega_0 t)$$

Let the open loop transfer function be

$G(s) = G(j\omega)$ in sinusoidal form.

The sinusoidal transfer function can be represented its magnitude and phase angle as

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

Substituting $\omega = \omega_0$ as per the input signal.

The output signal is

$$\boxed{c(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))}$$

* The amplitude of the output signal is obtained by multiplying the amplitude of the input with the magnitude of the sinusoidal transfer functer at $\omega = \omega_0$

* The phase angle of the output signal is obtained by adding the phase angle of the input signal with the phase of the ~~p~~ sinusoidal transfer function at $\omega = \omega_0$

Frequency domain Specifications

- Resonant Peak
- Resonant frequency
- Bandwidth

Frequency Domain Specifications

Consider the Second order system,

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Substitute $s = j\omega$ for the Sinusoidal transfer function

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)}$$

$$= \frac{\omega_n^2}{\cancel{s^2}(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + 2\xi\omega\omega_n j + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + j\frac{2\xi\omega}{\omega_n}\right)}$$

$$\therefore \boxed{T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\xi\omega}{\omega_n}\right)}}$$

Let $\frac{\omega}{\omega_n} = u$ the normalized frequency

substituting in the above equation,

$$\boxed{T(j\omega) = \frac{1}{(1-u^2)+j(2\xi u)}}$$

- Magnitude of the transfer function is

$$M = |T(j\omega)|$$

$$\boxed{M = \frac{1}{\sqrt{(1-u^2)+(2\xi u)^2}}}$$

- Phase of the transfer function is

$$\boxed{\angle T(j\omega) = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)}$$

- The steady state output of the system for a sinusoidal input of the unit magnitude and variable frequency ω is given by

$$c(t) = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)\right)$$

It can be seen from the above expressions that when

- $u=0$ then $M=1$ and $\phi=0$
- $u=1$ then $M=\frac{1}{2\xi}$ and $\phi=\frac{-\pi}{2}$
- $u\to\infty$ then $M\to 0$ and $\phi\to-\pi$

- Is the magnitude response monotonically decreasing from 1 to 0 (or) does it attain a maximum value and then decrease to 0.
- If it attains a maximum value at any frequency, then its derivative must be zero at that frequency.

$$\frac{dM}{du} = -\frac{1}{2}\frac{2(1-u^2)(-2u)+8\xi^2 u}{\left[(1-u^2)^2+(2\xi u)^2\right]^{3/2}} = 0$$

$$u^3-u+2\xi^2 u = 0$$

$$\boxed{u=0} \quad \text{(or)} \quad \boxed{u = \pm\sqrt{1-2\xi^2}}$$

$$\therefore \boxed{u = u_r = \sqrt{1-2\xi^2}}$$

(or)

$$\frac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$$

$$\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$$

This frequency where the magnitude becomes the maximum is known as the resonance frequency

Frequency Response Specifications:

- Substituting $u=u_r$, the maximum value of the response is known as Resonant peak $M=M_r$.

$$M_r = \frac{1}{\sqrt{(1-1+2\xi^2)^2+4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^4+4\xi^2-8\xi^4}}$$

$$\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$$

Phase angle at resonance peak

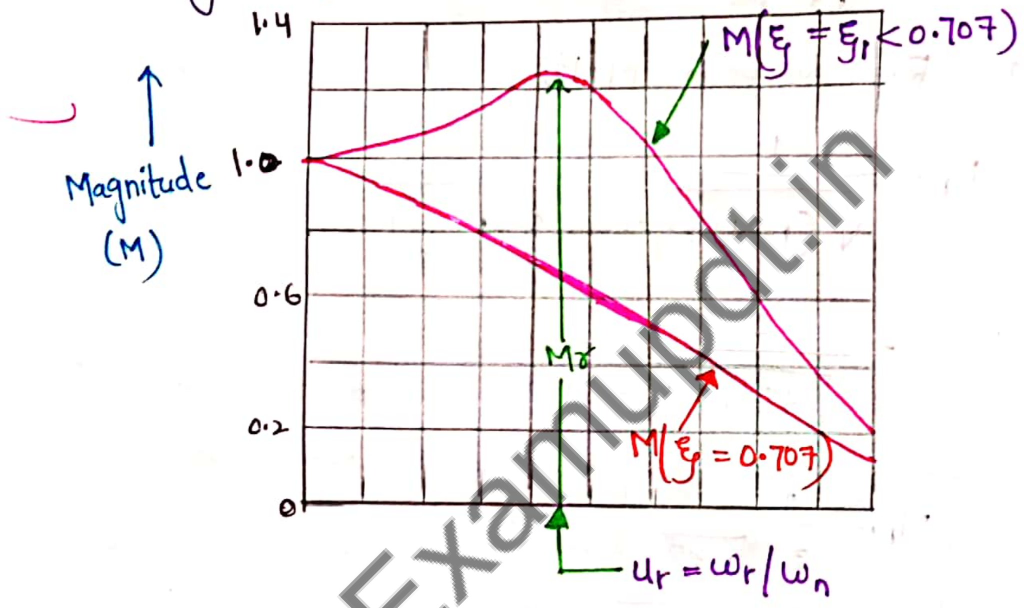
$$\phi_r = -\tan^{-1}\left(\frac{2\xi\sqrt{1-2\xi^2}}{1-1+2\xi^2}\right)$$

$$\boxed{\phi_r = -\tan^{-1}\left(\frac{\sqrt{1-2\xi^2}}{\xi}\right)}$$

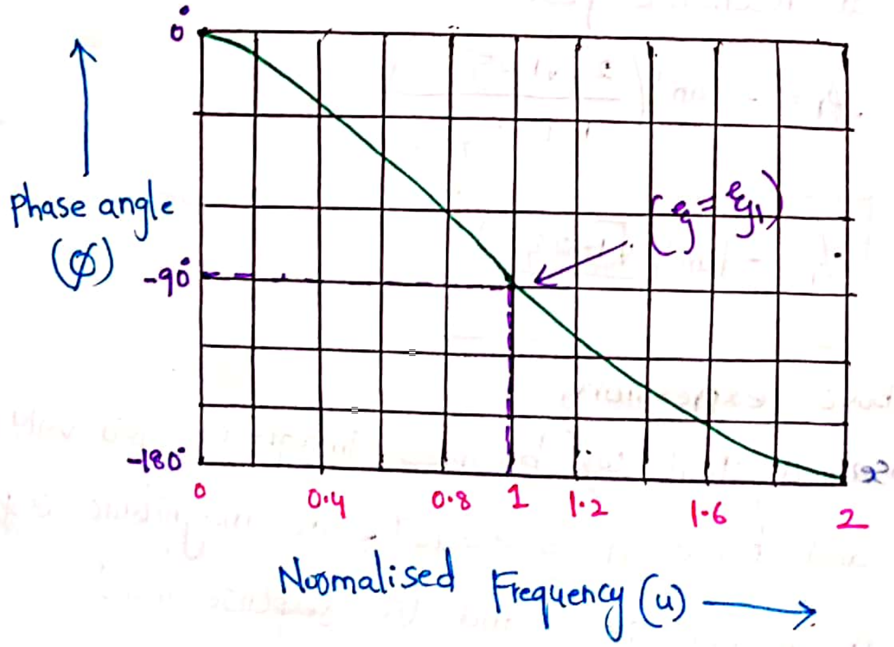
From the above expression,
it can be observed that ω_r becomes imaginary for values of $\xi > \frac{1}{\sqrt{2}}$ and hence if $\xi > 0.707$, the magnitude response doesn't have a resonant peak and the response monotonically reduces from 1 to 0.

- It therefore follows that $\xi > 0.707$, there is no resonant peak as such as the greatest value of M is 1.
- If $\xi = 0$, the magnitude response goes to infinity and this occurs at $\omega_r = \omega_n$, the natural frequency of the system.

From above expressions,
Resonant peak M_r is indicative of its damping ratio ξ and resonant frequency ω_r is indicative of ω_n for a given damping ratio, thus indicative of its speed of response (Settling Time).

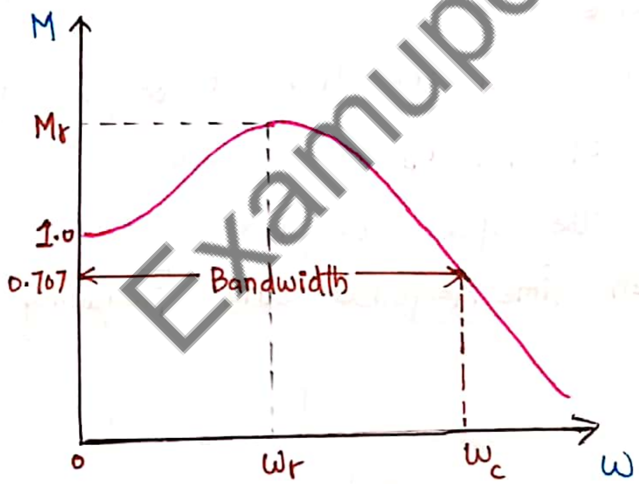


(a)



(b)

- In the magnitude response, it is to be noticed that for $\omega > \omega_r$, M decreases monotonically.
- The frequency at which M has its value to be $\frac{1}{\sqrt{2}}$ is of special significance and called the cut off frequency. The signal frequencies above cut-off are greatly attenuated in passing through a system.
- At this frequency, the magnitude will be $20 \log \frac{1}{\sqrt{2}} = -3dB$.
- For feedback control systems, the range of frequencies over which M is equal to (or) greater than $\frac{1}{\sqrt{2}}$ is defined as bandwidth ω_b.
- As M=1 for u=0, control systems are considered as low pass filters and the frequency at which the magnitude to -3dB is known as the bandwidth.



An expression for bandwidth can be obtained by equating M=0.707 at $u=u_0$ in the expression for magnitude.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + 4\xi^2 u_b^2}}$$

$$2 = (1-u_b^2)^2 + 4\xi^2 u_b^2$$

Let $u_b^2 = x$,

$$2 = (1-x)^2 + (2\xi)^2 x$$

$$x^2+(4\xi^2-2)x-1=0$$

$$x=\frac{-(4\xi^2-2)\pm\sqrt{(4\xi^2-2)^2+4}}{2}$$

$$=(1-2\xi^2)+\sqrt{(2\xi^2-1)^2+1}$$

$$x=1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}$$

$$u_b^2=1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}$$

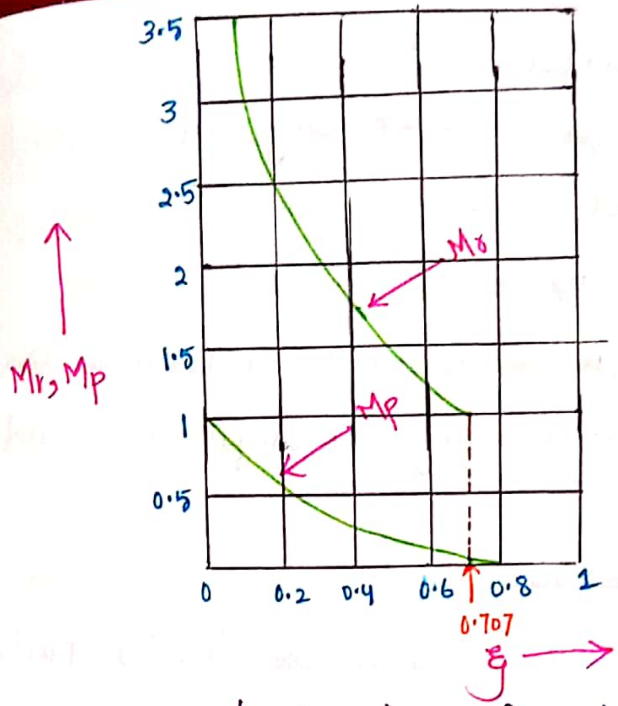
$$u_b=\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}$$

$$\Rightarrow \boxed{\omega_b=\omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}}$$

- Bandwidth of a control system indicates the noise filtering characteristic of the system i.e., more the bandwidth, more susceptible to noise.
- It is also a measure of the transient response properties. For a given value of ξ, ω_b is measure of ω_n, so more bandwidth, higher the speed response.

Correlation between Time Response and Frequency response:

Time Response	Frequency Response
• Time domain specifications obtained for second order system subjected to step input	• In frequency domain, system subjected to constant amplitude, variable frequency sinusoidal signal
• Peak overshoot $M_p=e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$; $0<\xi<1$	• Resonant peak $M_r=\frac{1}{2\xi\sqrt{1-\xi^2}}$; $\xi<0.707$
• Damped frequency of oscillations $\omega_d=\omega_n\sqrt{1-\xi^2}$	• Resonant Frequency $\omega_r=\omega_n\sqrt{1-2\xi^2}$
• Settling time $t_s=\frac{4}{\xi\omega_n}$	• Band width $\omega_b=\omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}$



- Comparision of expressions for M_p and M_r shows that both are dependent on damping factor ξ only.
- Given M_p, the resonant peak M_r can be evaluated provided ξ is less than 0.707.
- This condition is usually satisfied by many practical control systems as ξ is seldom greater than 0.707.
- Thus the resonant peak M_r and peak overshoot are well correlated.
- The expressions for time domain specification of damped natural frequency and the frequency domain specification of resonance frequency.
- For a given damping factor ξ, the ratio $\frac{\omega_r}{\omega_d} = \sqrt{\frac{1-2\xi^2}{1-\xi^2}}$ is fixed and given the frequency domain specification the time domain specifications and vice-versa can be easily obtained.
- The speed of the response is indicated by settling time in time domain. In frequency domain, bandwidth is an indication of the speed of the response.
- There is a perfect correlation between the time domain and the frequency domain performance measures and given one,

other can be easily obtained.

- Correlation is satisfied for $\xi < 0.707$ only and is usually satisfied by most control systems.

Advantages of Frequency Response

- Signal generators and precise measuring instruments are readily available for various ranges of frequencies and amplitudes.
- Ease and accuracy of measurements.
- Whenever it is not possible to obtain the transfer function of a system through analytical methods, the frequency response can be used to compute the same
- Design and parameter adjustment of the open loop system for a desired closed loop performance is carried out more easily in frequency response.
- Effects of noise disturbance and parameter variations can be easily visualized.
- The necessary transient response can be obtained from the frequency response through the correlated parameters

Disadvantages of Frequency Response.

- For systems with very large time constants, the frequency response test is cumbersome to perform as the time required to obtained the steady state response is very large.
- It cannot be performed on non interruptible systems.

Frequency Domain plots.

- Bode plot
- Polar Plot (Polar Graph).
- Nyquist plot

Bode plot:

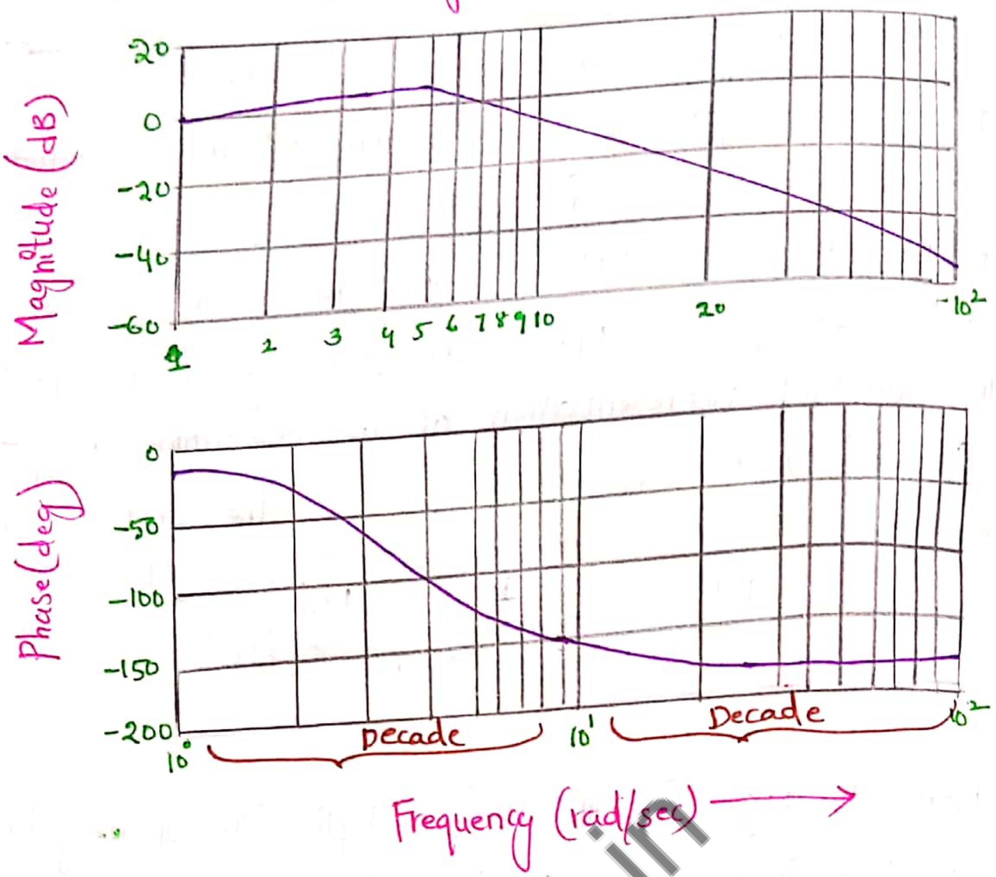
- A Bode diagram Consists of two graphs:
  - * One is a plot of the logarithm of the magnitude of a Sinusoidal transfer function.
  - * The other is a plot of the phase angle.
  - * Both are plotted against the frequency on a logarithmic scale.
- The standard representation of the logarithmic magnitude of $G(j\omega)H(j\omega)$ is $20\log_{10}|G(j\omega)H(j\omega)|$. The unit is dB.

$$\boxed{G(j\omega) = \frac{K(1+j\omega i_1)(1+j\omega i_2)}{20\log(1+j\omega i_3)(1+j\omega i_4)}}$$

Advantages of Logarithmic plots:

- Main advantage is that the multiplication of magnitudes is Converted into addition due to logarithm.
- It is very easy to conduct bode plots using asymptotic approximation.
- It allows study of impact of individual poles and zero.
- Gain and phase margins can be obtained without tedious Calculation.
- It is very easy to obtain frequency domain specifications from Bode plots.
- Low and high frequency characteristics can be illustrated in one diagram.
- It is easy to assess and understand the relative stability with Bode plots.

Bode Diagram



$1 \rightarrow 10$
$10 \rightarrow 100$ } Decade (multiplication by 10)
$100 \rightarrow 1000$

$1 \rightarrow 2$
$2 \rightarrow 4$ } octave (Multiplication by 2)
$4 \rightarrow 8$
$3 \rightarrow 6$

Something multiply (or) divide by 2 – octave
Something multiply (or) divide by 10 – Decade
Correction to approximate bode plot is done at octave & decade.

Basic Factors of a Transfer Function

The basic factors that very frequently occur in a arbitary transfer function are

1. Gain (K)
2. Integral and Derivative Factors $(j\omega)^{\pm 1}$
3. First order Factors $(j\omega T+1)^{\pm 1}$
4. Quadratic Factors

$$\left[1+2\xi(j\omega/j\omega_n)+(j\omega/\omega_n)^2\right]^{\mp 1}$$

$$\boxed{G(s)=\frac{20(3s+1)}{s(s+1)(s^2+5s+2)}}$$

$$G(j\omega)=\frac{K(1+j\omega i_{z_1})(1+j\omega i_{z_2})}{j\omega(1+j\omega i_{p_1})(1+j\omega i_{p_2})}$$

$$G(j\omega)=\frac{K\sqrt{1+\omega^2 i_{z_1}^2}\ \sqrt{1+\omega^2 i_{z_2}^2}}{\omega\sqrt{1+\omega^2 i_{p_1}^2}\ \sqrt{1+\omega^2 i_{p_2}^2}}$$

$$|G(j\omega)|=20\log\left|\frac{K\sqrt{1+\omega^2 i_{z_1}^2}\ \sqrt{1+\omega^2 i_{z_2}^2}}{(j\omega)(1+j\omega i_{p_1})(1+j\omega i_{p_2})}\right|$$

$$\therefore \boxed{|G(j\omega)|=20\log K+20\log|1+j\omega i_{z_1}|+20\log|1+j\omega i_{z_2}|-20\log|j\omega|-20\log|1+j\omega i_{p_1}|-20\log|1+j\omega i_{p_2}|}$$

Basic factors,

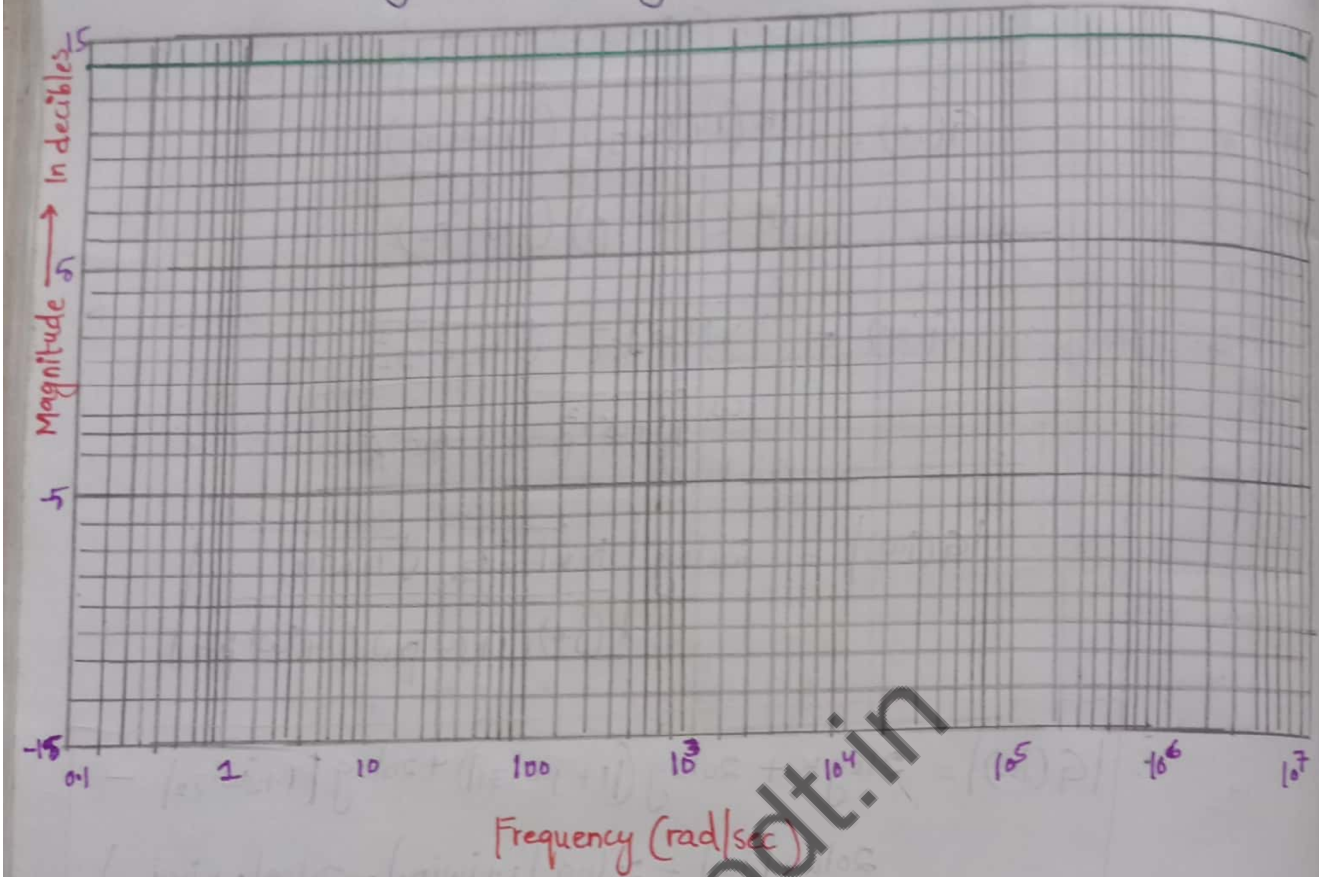
K, $(1+\omega i_{z_1})^n$, $\frac{1}{(1+j\omega i_{p_1})^n}$

1. Gain (K).

- The log-magnitude curve for a constant gain K is a horizontal straight line at the magnitude of $20\log(K)$ decibels.
- The phase angle of the gain k is zero.
- The effect of varying the gain K in the transfer function is that it raises (or) lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

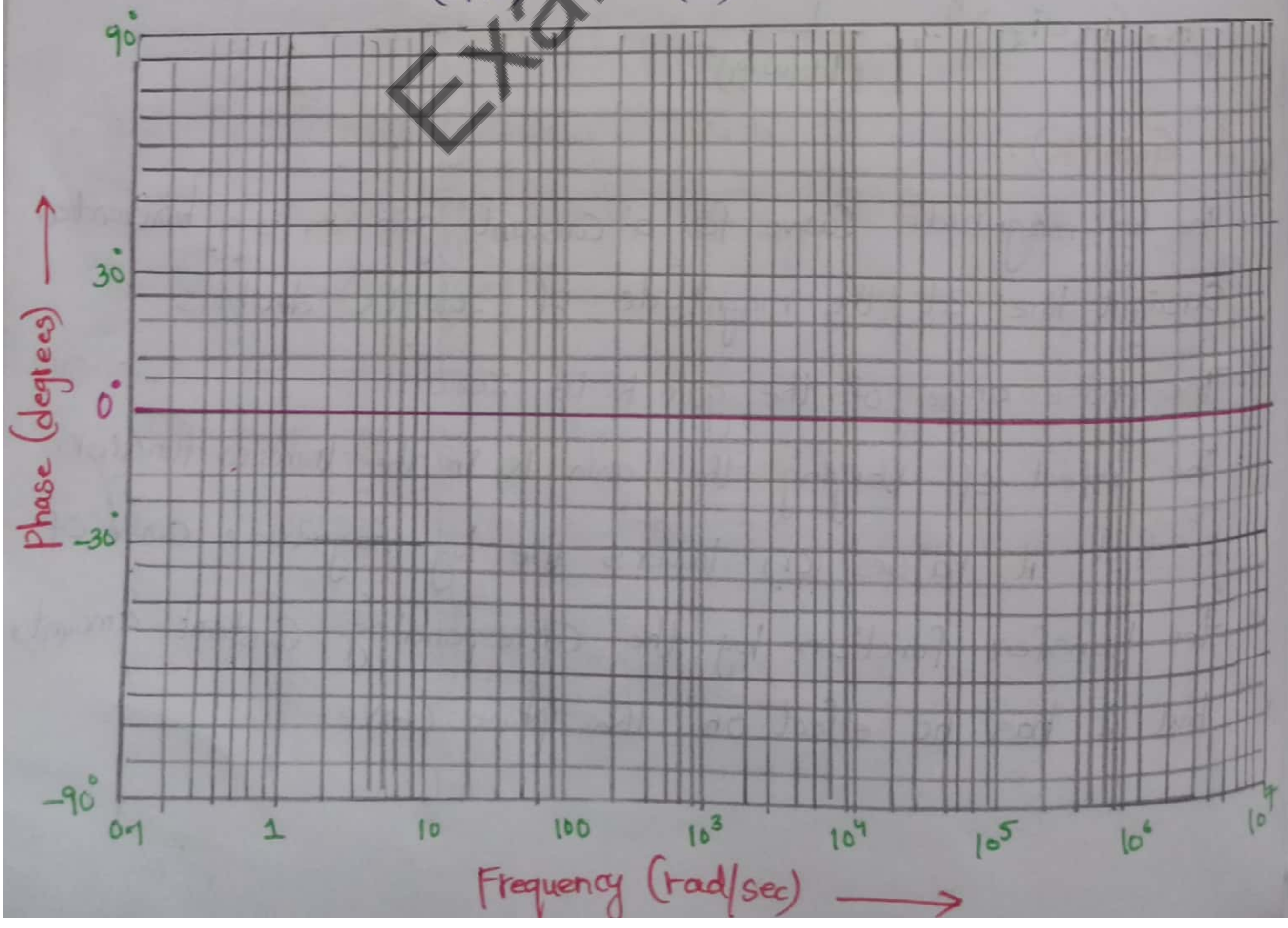
If $K=5$, then

$$20\log(K) = 20\log(5) = 14\,db$$



If $K=5$, then

$$\phi = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{0}{5}\right) = 0$$



2. Integral and Derivative Factors $(j\omega)^{\pm 1}$

Derivative Factor,

$$G(s) = s$$

Where $s = j\omega$

Magnitude,

$$|G(j\omega)| = 20\log(\omega)$$

ω	0.1	0.2	0.4	0.5	0.7	0.8	0.9	1
db	−20	−14	−8	−6	−3	−2	−1	0

slope = 6b/octave

$$\text{slope} = \frac{20\text{db}}{\text{decade}}$$

Phase,

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = 90^\circ$$

When expressed in decibels, the reciprocal of a number differs from its value only in sign; that is, for the number N.

$$20\log N = -20\log\left(\frac{1}{N}\right)$$

Therefore, for integral factor the slope of the magnitude line would be same but with opposite sign (i.e −6db/octave (or) −20db/decade).

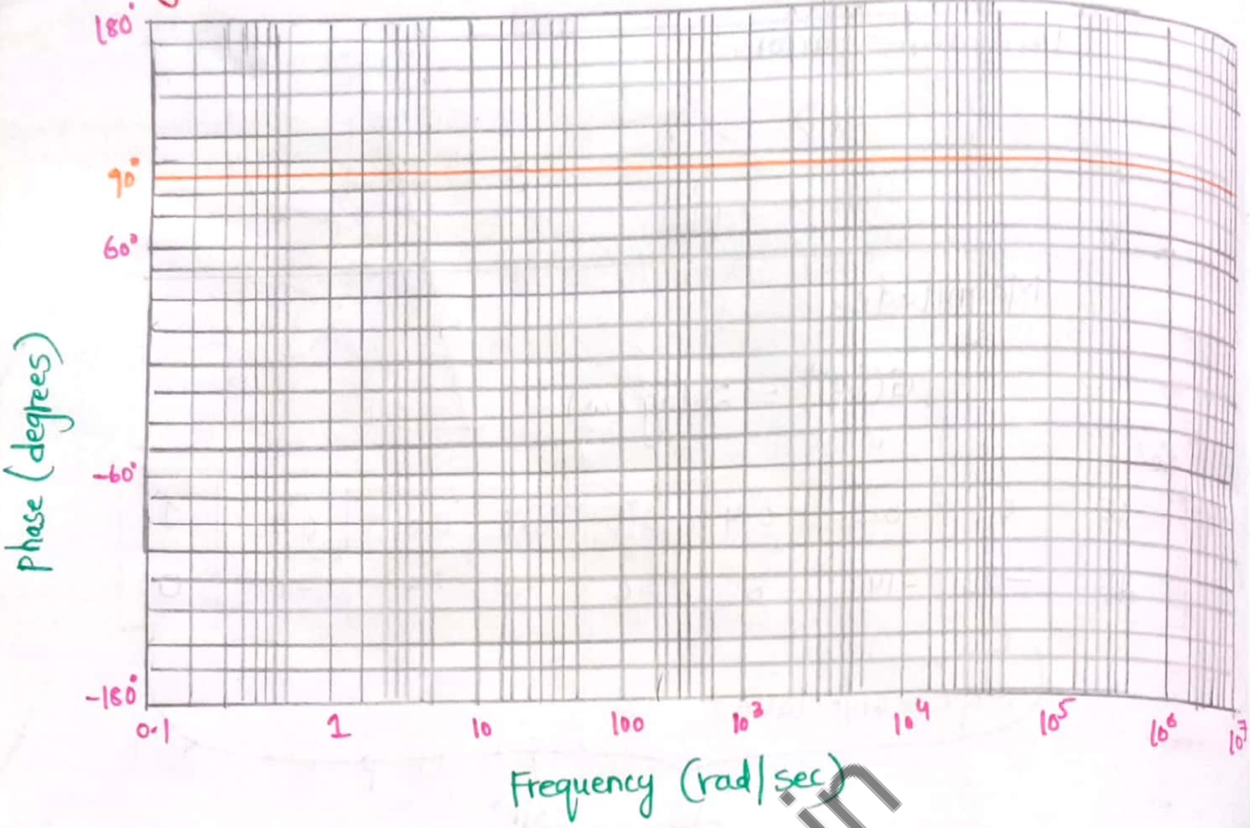
Magnitude,

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = -20\log(\omega)$$

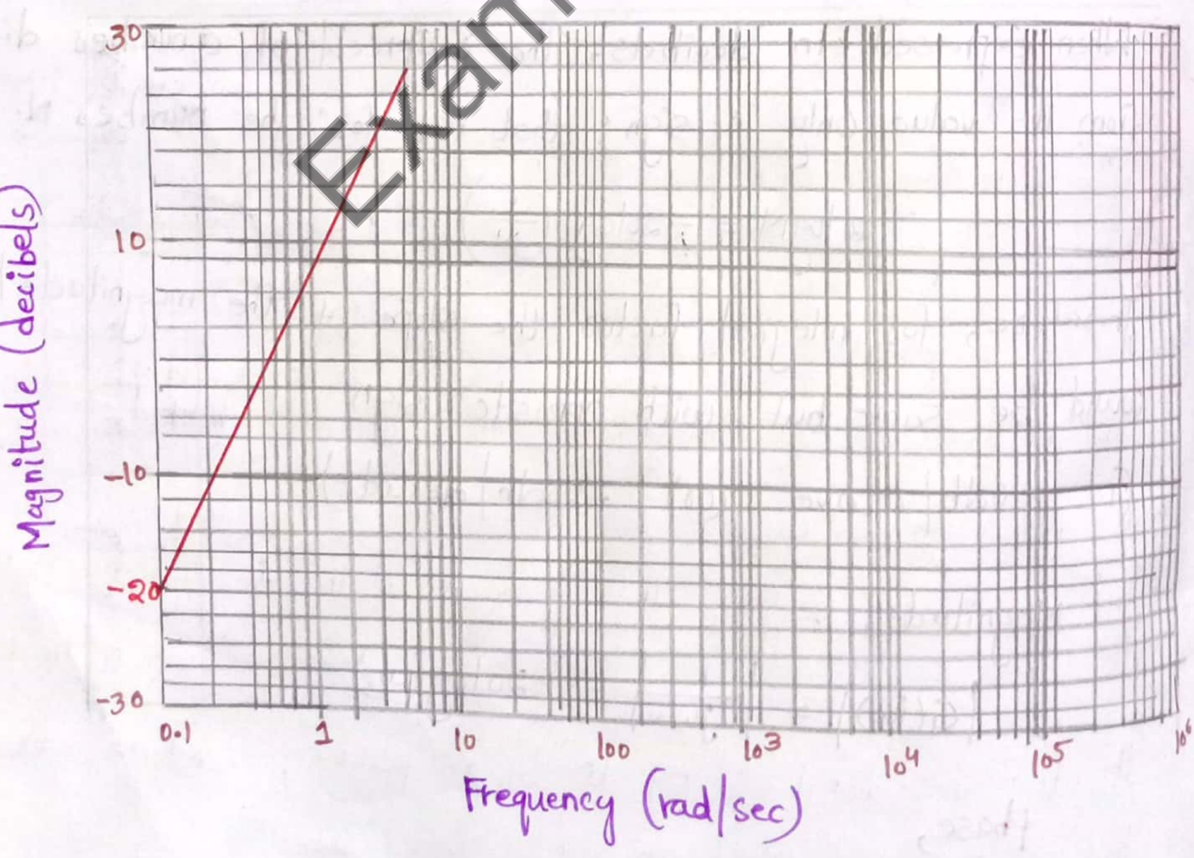
Phase,

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) = -90^\circ$$

Magnitude = 20db/decade & phase = 90°

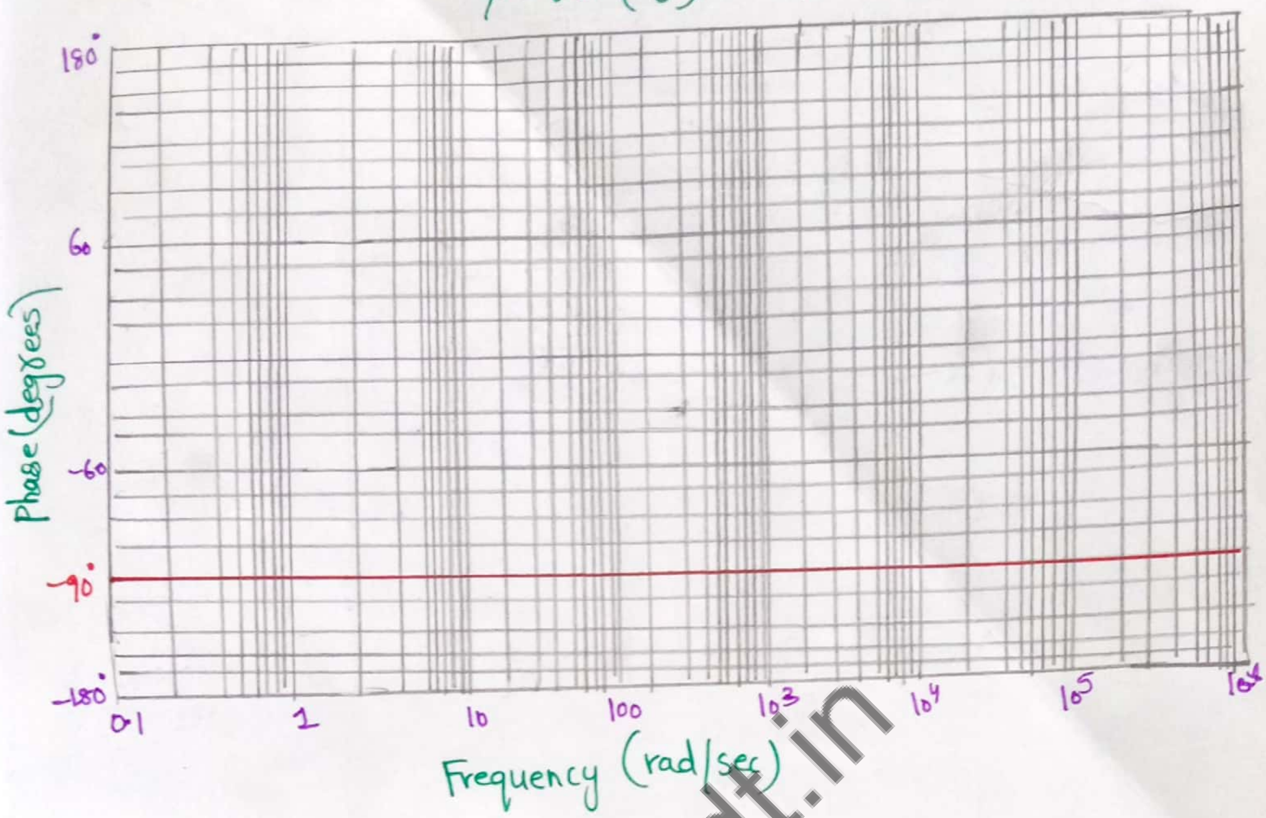


20db/decade

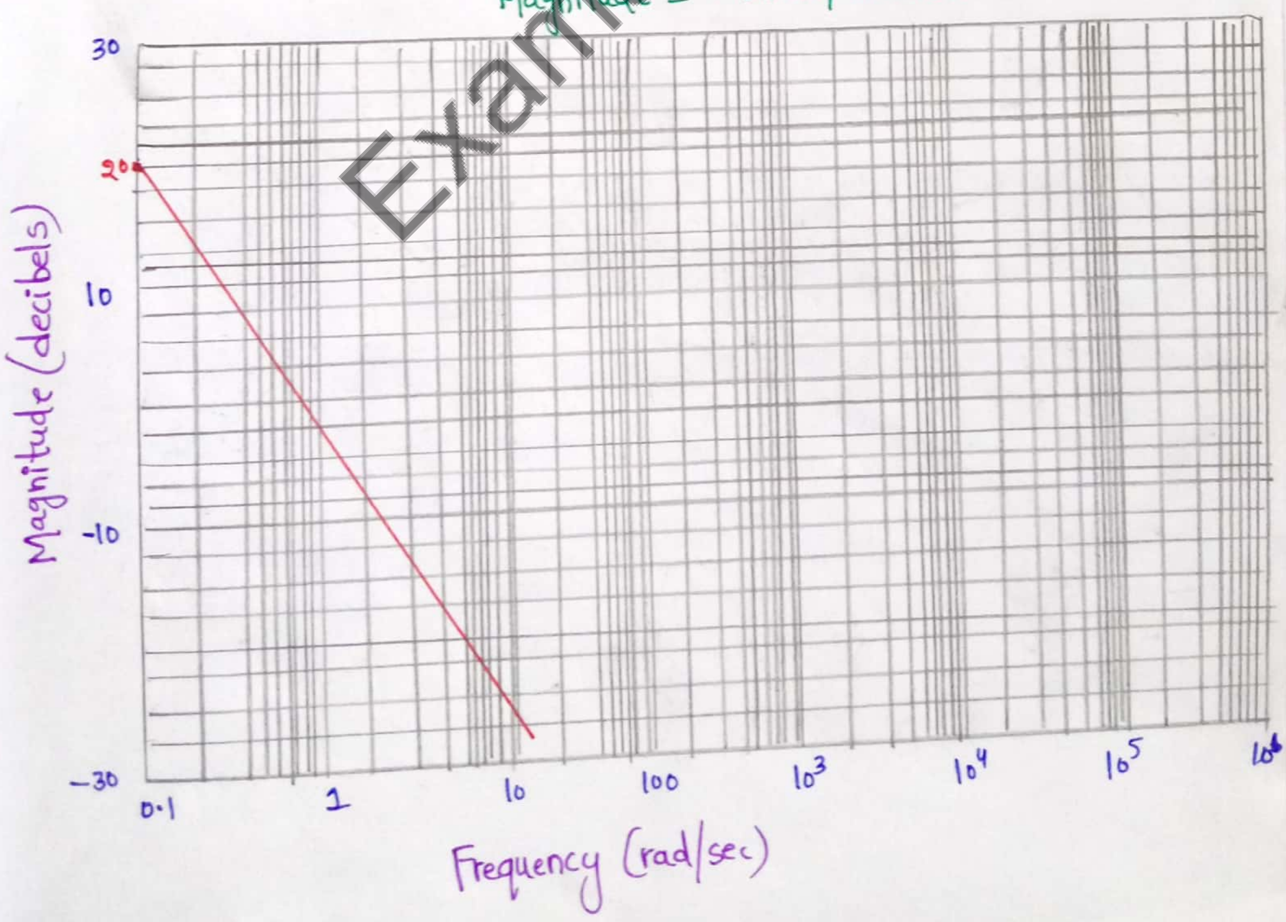


Magnitude = -20db/decade and phase = -90°

$\phi = \tan^{-1}\left(\frac{\omega}{0}\right) = -90^\circ$



Magnitude = - 20db/decade



3. First order Factors $(j\omega T+1)^{\pm 1}$

First order Factors $(j\omega T+1)$

$$M(\omega) = 20\log(|1+j\omega T|)$$

$$M(\omega) = 20\log\left(\sqrt{1+\omega^2T^2}\right)$$

For low frequencies $\omega << \frac{1}{T}$

$$\omega << \frac{1}{T}$$

$$\omega T << 1$$

$$\boxed{M(\omega) = 20\log(1) = 0}$$

For high frequencies $\omega >> \frac{1}{T}$

$$\omega >> \frac{1}{T}$$

$$\omega T >> 1$$

$$M(\omega) = 20\log\sqrt{1+\omega^2T^2}$$

$$M(\omega) = 20\log\sqrt{\omega^2T^2}$$

$$\boxed{M(\omega) = 20\log(\omega T)}$$

$$\phi(\omega) = \tan^{-1}(\omega T)$$

When $\omega = 0$ then $\phi(\omega) = \tan^{-1}(0) = 0^\circ$

When $\omega = \frac{1}{T}$ then $\phi(\omega) = \tan^{-1}(1) = 45^\circ$

When $\omega = \infty$ then $\phi(\omega) = \tan^{-1}(\infty) = 90^\circ$

$$G(s) = (s+3) = \left(\frac{1}{3}s+1\right)$$

$$\uparrow \frac{1}{T} \qquad \uparrow T$$

$$G(j\omega) = \left(\frac{1}{3}j\omega+1\right)$$

$$T = \frac{1}{3}$$

$$\omega = \frac{1}{T} = 3$$

If $\omega = 3$

$$M = 20\log\sqrt{1+\omega^2T^2}$$

$$= 20\log\sqrt{1+(3)^2\left(\frac{1}{3}\right)^2}$$

$$= 20\log\sqrt{2}$$

$$M = 3.01\,dB$$

If $\omega = 1.5$

$$M = 20\log\sqrt{1+\omega^2T^2}$$

$$= 20\log\sqrt{1+\left(\frac{1}{2}\right)^2(1)}$$

$$M = 1\,db$$

If $\omega = 6$

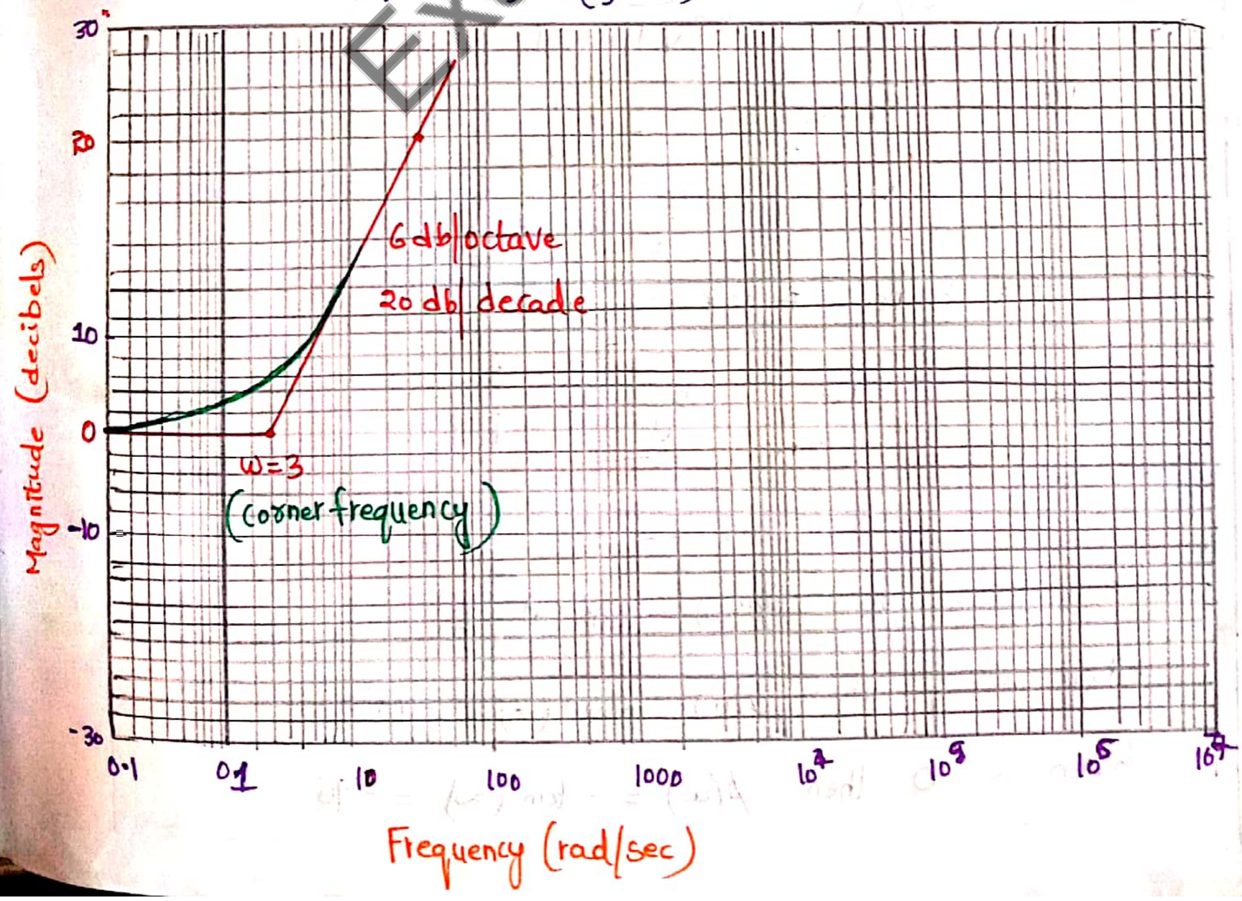
$$M = 20\log\sqrt{1+4}$$

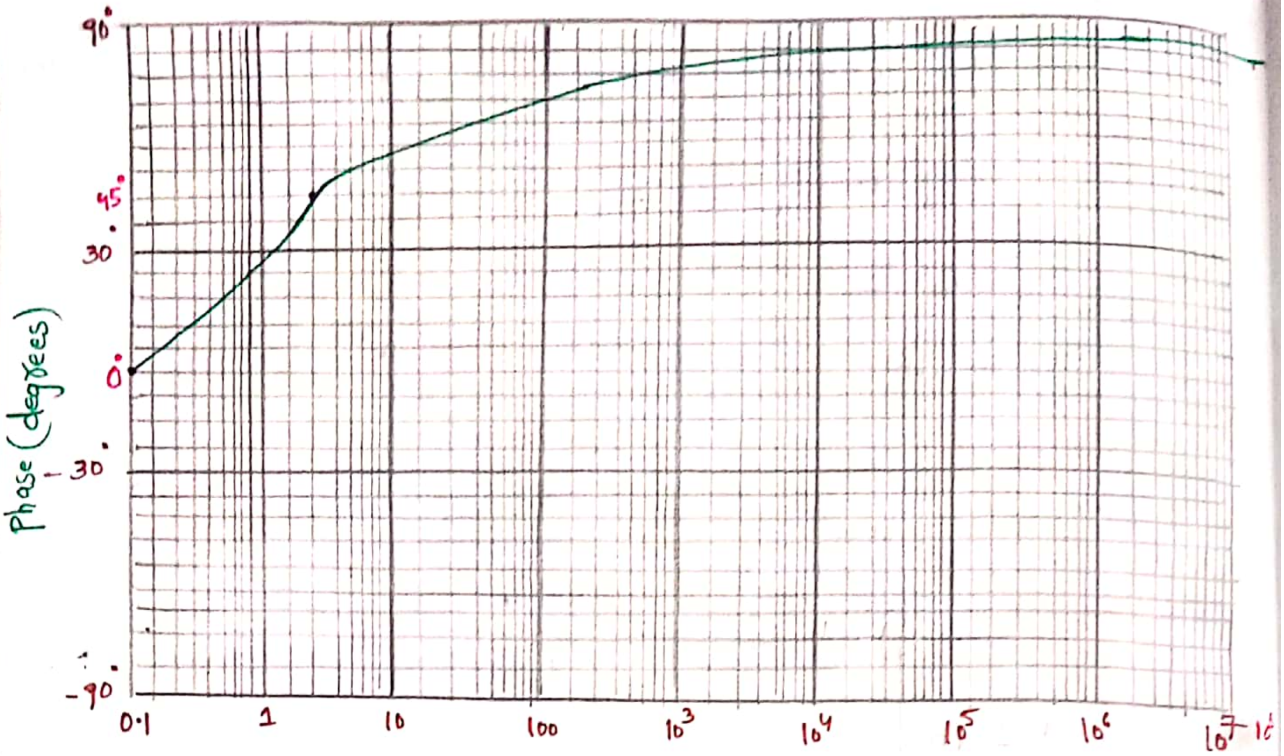
$$M = 20\log\sqrt{5}$$

$$M = 6.98$$

$$M \simeq 7$$

$$G(s) = (s+3) = \left(\frac{1}{3}s+1\right)$$





The graphs of magnitude (decibels) and phase (degrees versus frequency is drawn.

First order Factor $(j\omega T+1)^{-1}$

$$M(\omega) = -20\log(|1+j\omega T|)$$

$$M(\omega) = -20\log\sqrt{1+\omega^2T^2}$$

For low frequencies $\omega << \frac{1}{T}$

$$\boxed{M(\omega) = -20\log(1) = 0}$$

For higher frequencies $\omega >> \frac{1}{T}$

$$\boxed{M(\omega) = -20\log(\omega T)}$$

$$\phi = -\tan^{-1}(\omega T)$$

When $\omega = 0$ then $\phi(\omega) = \tan^{-1}(0) = 0°$

When $\omega = \frac{1}{T}$ then $\phi(\omega) = -\tan^{-1}(1) = -45°$

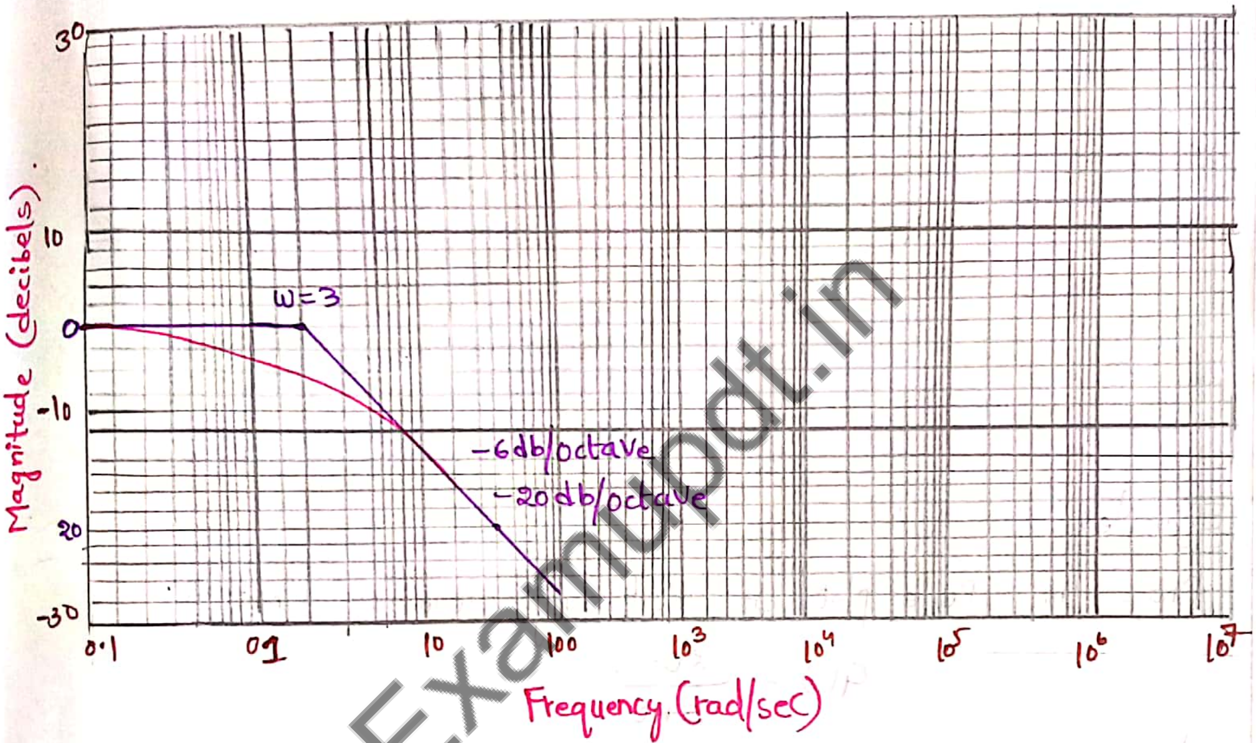
When $\omega = \infty$ then $\phi(\omega) = -\tan^{-1}(\infty) = -90°$

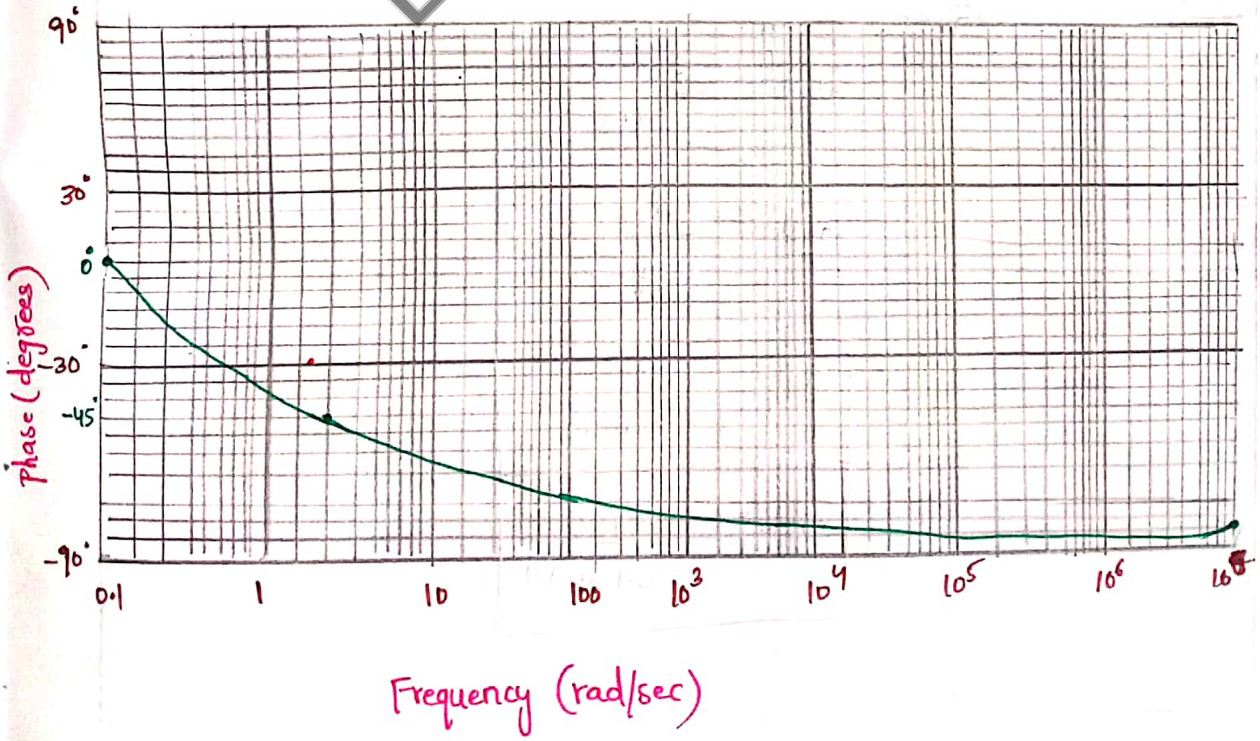
$$G(s) = \frac{1}{s+3}$$

$$G(s) = \frac{1}{3\left(\frac{1}{3}s+1\right)}$$

$$G(j\omega) = \frac{\frac{1}{3}}{\left(1+j\omega\frac{1}{3}\right)}$$

$$G(s) = \frac{1}{s+3}$$





Example 1:

Draw the Bode plote of the following Transfer function

$$G(s) = \frac{20s}{s+10}$$

Sol: Given, $G(s) = \dfrac{20s}{s+10}$

$$= \frac{20s}{10\left(1+\frac{1}{10}s\right)}$$

$$\boxed{\begin{aligned} G(s) &= \frac{2s}{1+0.1s} \\ G(j\omega) &= \frac{2j\omega}{1+0.1j\omega} \end{aligned}}$$

The transfer function Contains

1. Gain factor $(K=2)$
2. Derivative factor (s)
3. 1^{st} order factor in denominator $(0.1s+1)^{-1}$

1. Gain factor $(K=2)$

$$|K|_{db} = 20\log(2) = 6db$$

2. Derivative factor(s)

$$|s|_{db} = 20\log(\omega) = 20db/decade$$

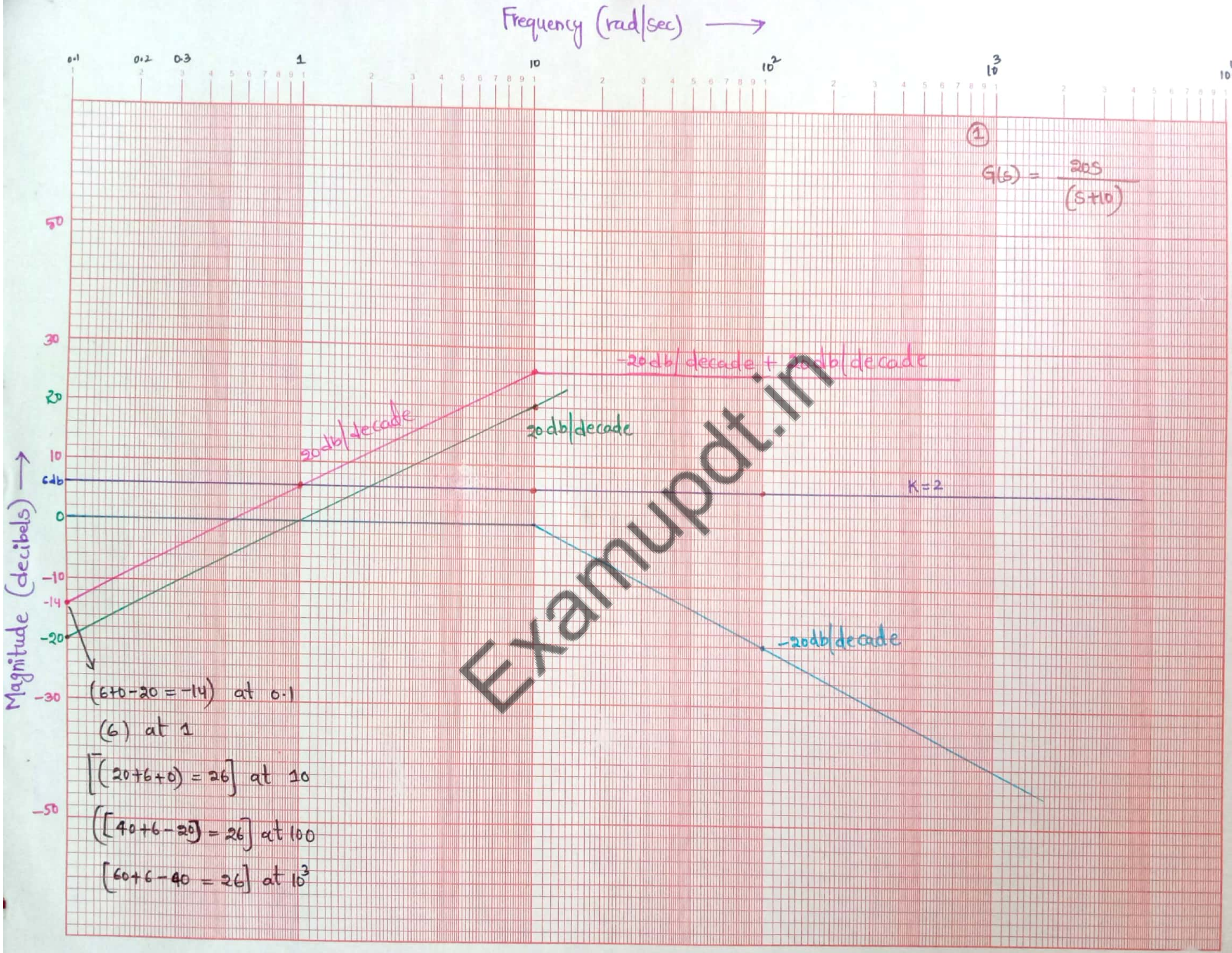
3. 1^{st} order Factor in denominator $(0.1s+1)^{-1}$

When $\omega << 10$,

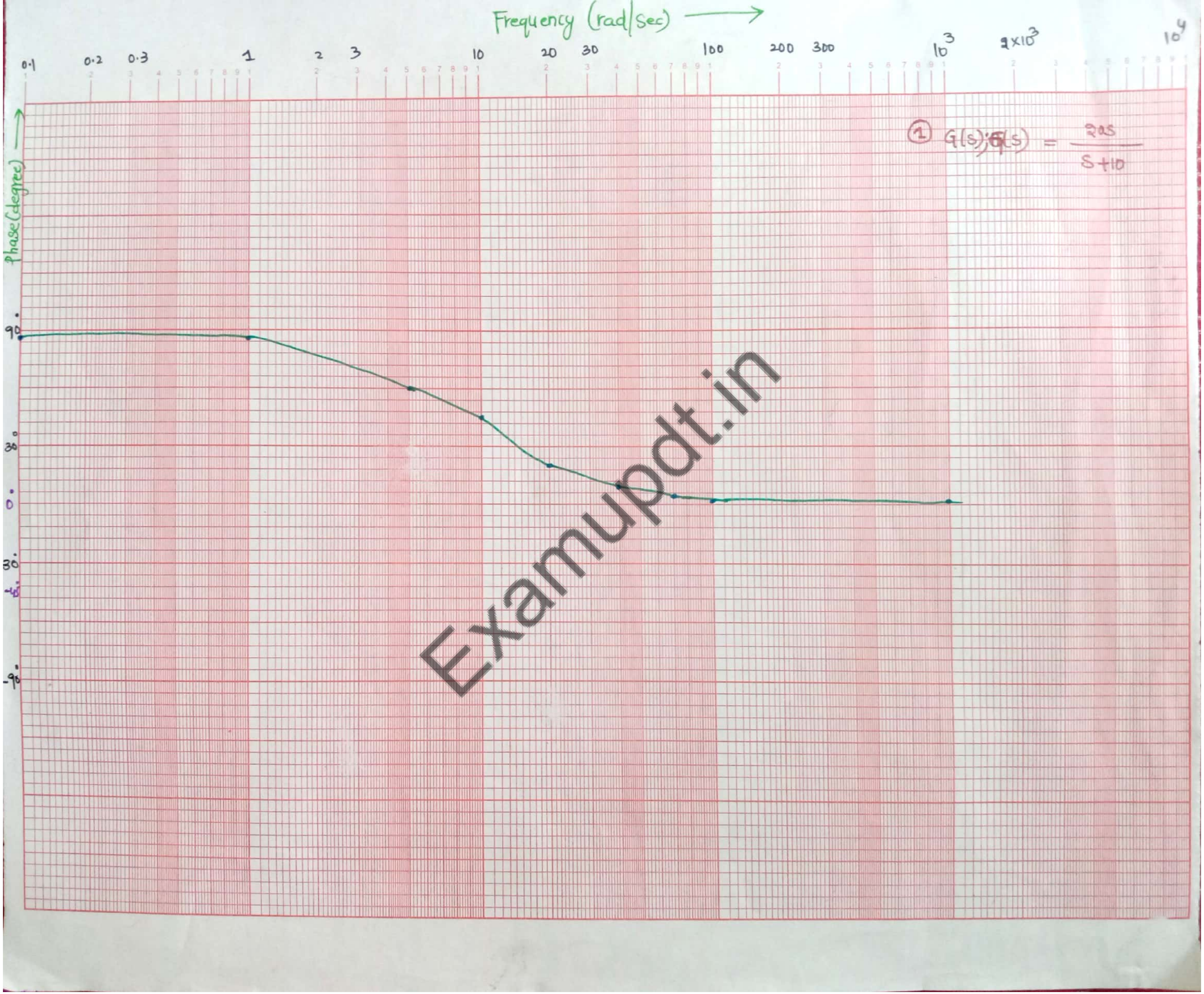
$$\left|\frac{1}{0.1j\omega+1}\right|_{db} = -20\log(1) = 0$$

When $\omega >> 10$,

$$\left|\frac{1}{1+0.1j\omega}\right|_{db} = -2\log(0.1\omega) = -20\,db/decade$$



The 3 colours (Green, blue, Thick blue) are seperate basic factors ; Pink → overall plot.



Frequency (rad/sec)
0.1
0.2
0.3
1
2
3
10
20
30
100
200
300
10^3
2×10^3
10^4
Phase (degree)
90°
30°
0°
-30°
-40°
-90°
① $G(s)H(s) = \frac{20s}{s+10}$

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$$G(j\omega) = \frac{2j\omega}{0.1j\omega+1}$$

$$\angle G(j\omega) = \angle 2 + \angle j\omega - \angle(0.1j\omega+1)$$

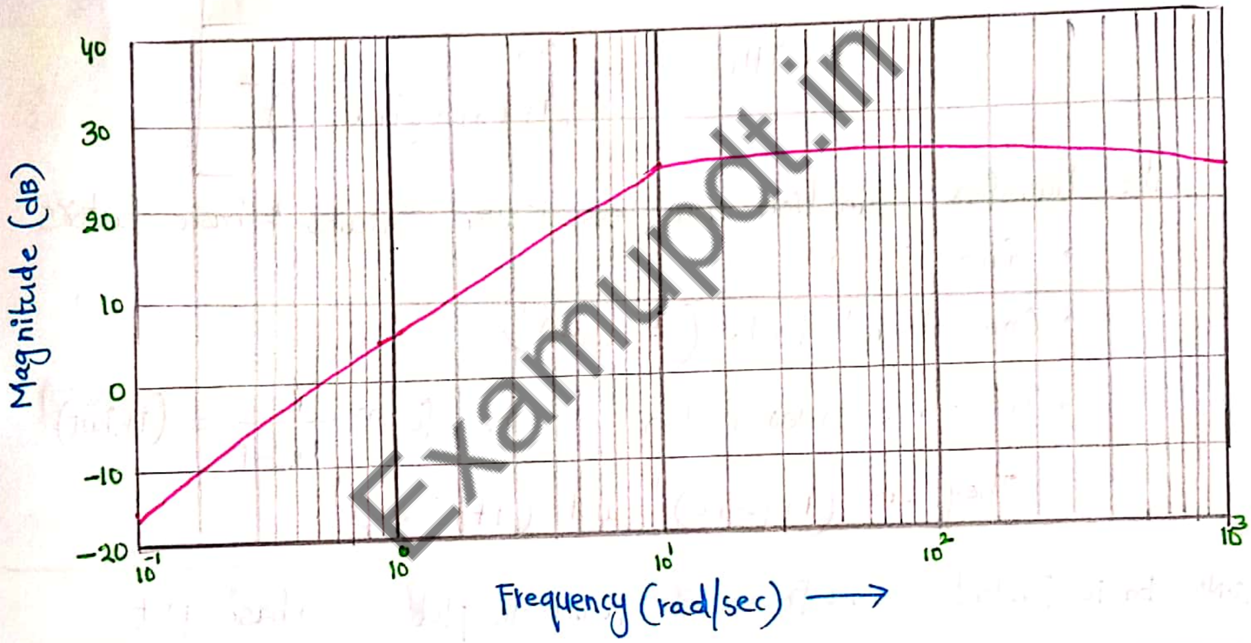
$$\angle G(j\omega) = \tan^{-1}\left(\frac{0}{2}\right) + \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}(0.1\omega)$$

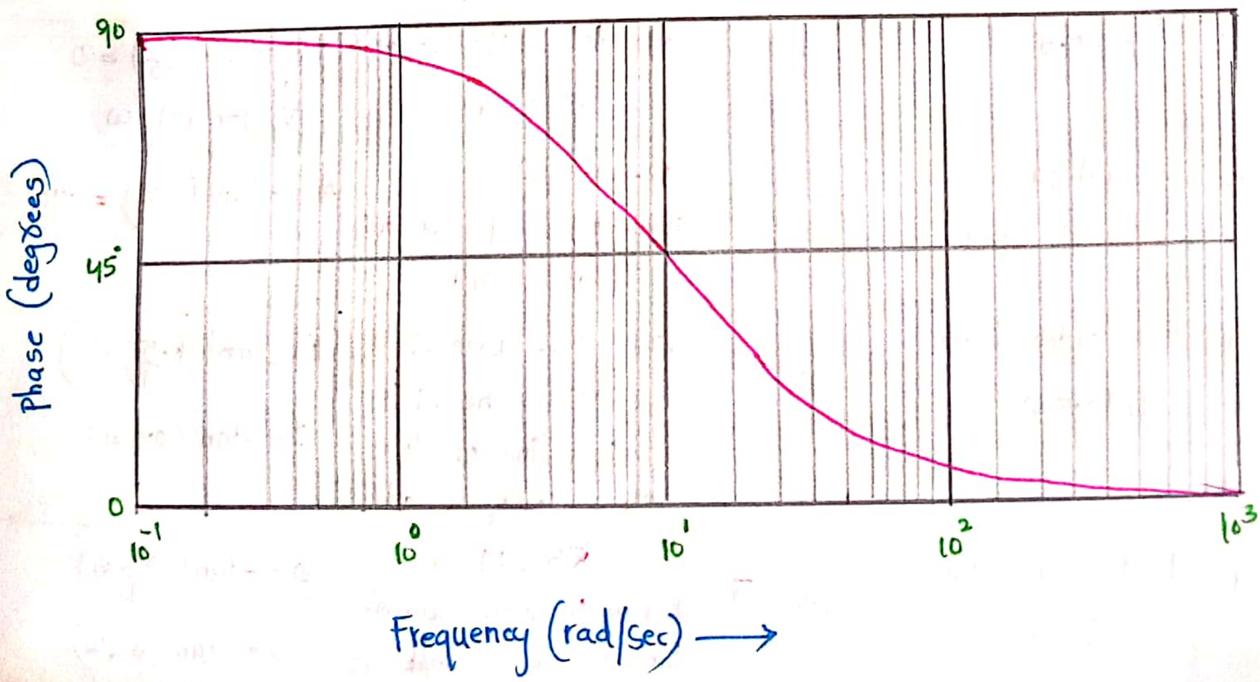
$$\boxed{\angle G(j\omega) = 90^\circ - \tan^{-1}(0.1\omega)}$$

ω	0.1	1	5	10	20	40	70	100	1000	∞
$\phi(\omega)$	89.4	84.2	63.4	45	26.5	14	8	5.7	0.5	0

Matrix laboratory

Bode Diagram:





Example-B:

$$G(s)H(s) = \frac{5}{s(s+2)(s+5)}$$

Sol: Converting the transfer function into a time constant and Sinusoidal form

$$G(s)H(s) = \frac{5}{s(s+2)(s+5)}$$

$$G(s)H(s) = \frac{5}{s\, 2(\frac{s}{2}+1)\, 5(1+\frac{s}{5})}$$

$$\boxed{G(s)H(s) = \frac{0.5}{s(1+0.5s)(1+0.2s)}}$$

$$\boxed{G(j\omega)H(j\omega) = \frac{0.5}{j\omega(1+0.5j\omega)(1+0.2j\omega)}}$$

The transfer function now essentially contains 4 basic factors

- Gain $(K=0.5)$
- One integral factor $\left(\frac{1}{s} = (j\omega)^{-1}\right)$
- Two first order factor of the form $\frac{1}{1+j\omega T} = (1+j\omega T)^{-1}$

They are $(1+j0.5\omega)^{-1}$ and $(1+j0.2\omega)^{-1}$.

S.No	Basic Factor	Corner Frequency	Magnitude plot	Phase plot
1	$K=0.5$	–	Constant at $M=20\log(0.5)$ $M=-6dB$	$\phi=\tan^{-1}\left(\frac{0}{0.5}\right)=0$ $\forall$ (For all) ω
2	Integral factor $\left(\frac{1}{s}\right)$ or $(j\omega)^{-1}$	–	$M=-20\log(\omega)$ straight line of slope at -20db/decade	$\phi=-\tan^{-1}\left(\frac{\omega}{0}\right)=-90^\circ$
3.	First order factor $(1+j0.5\omega)^{-1}$	$\omega=\frac{1}{0.5}=2$	For $\omega<2$ then $M=0$ For $\omega>2$ the M is straight line of slope at -20db/decade	$\phi=-\tan^{-1}\left(\left(\frac{0.5}{1}\right)\omega\right)$ $\phi=-\tan^{-1}(0.5\omega)$
4	First order factor $(1+j0.2\omega)^{-1}$	$\omega=\frac{1}{0.5}=5$	For $\omega\leq5$, $M=0$ For $\omega>5$ then straight line of slope -20db/decade	$\phi=-\tan^{-1}\left(\frac{0.2}{1}\omega\right)$ $\phi=-\tan^{-1}(0.2\omega)$

Magnitude plot

Factor	Starting point/ Corner Frequency	Magnitude	Slope	Net slope (changes)
$K=0.5$	–	$-6dB$ Constant	0	0
$(j\omega)^{-1}$	For $\omega=1$ $M=-20\log\omega$ $M=0$	$-6dB$	$-20db/decade$	$-20db/decade$
$(1+j0.5\omega)^{-1}$	$\omega<2$, $M=0$ $\omega>2$		$-20\ db/decade$	$-40\ db/decade$
$(1+j0.2\omega)^{-1}$	$\omega<5$, $M=0$ $\omega>5$		$-20\ db/decade$	$-60\ db/decade$

$$\frac{K}{(j\omega)^1} = \frac{0.5}{j\omega}$$

$$M = 20\log(0.5) - 20\log\omega$$

$$M = -20\log\omega + 20\log(0.5)$$

$$\boxed{M = -20\log\omega - 6}$$

$$y = mx + c$$

Phase plot

$$G(j\omega) = \frac{0.5}{j\omega(1+0.5j\omega)(1+j0.2\omega)}$$

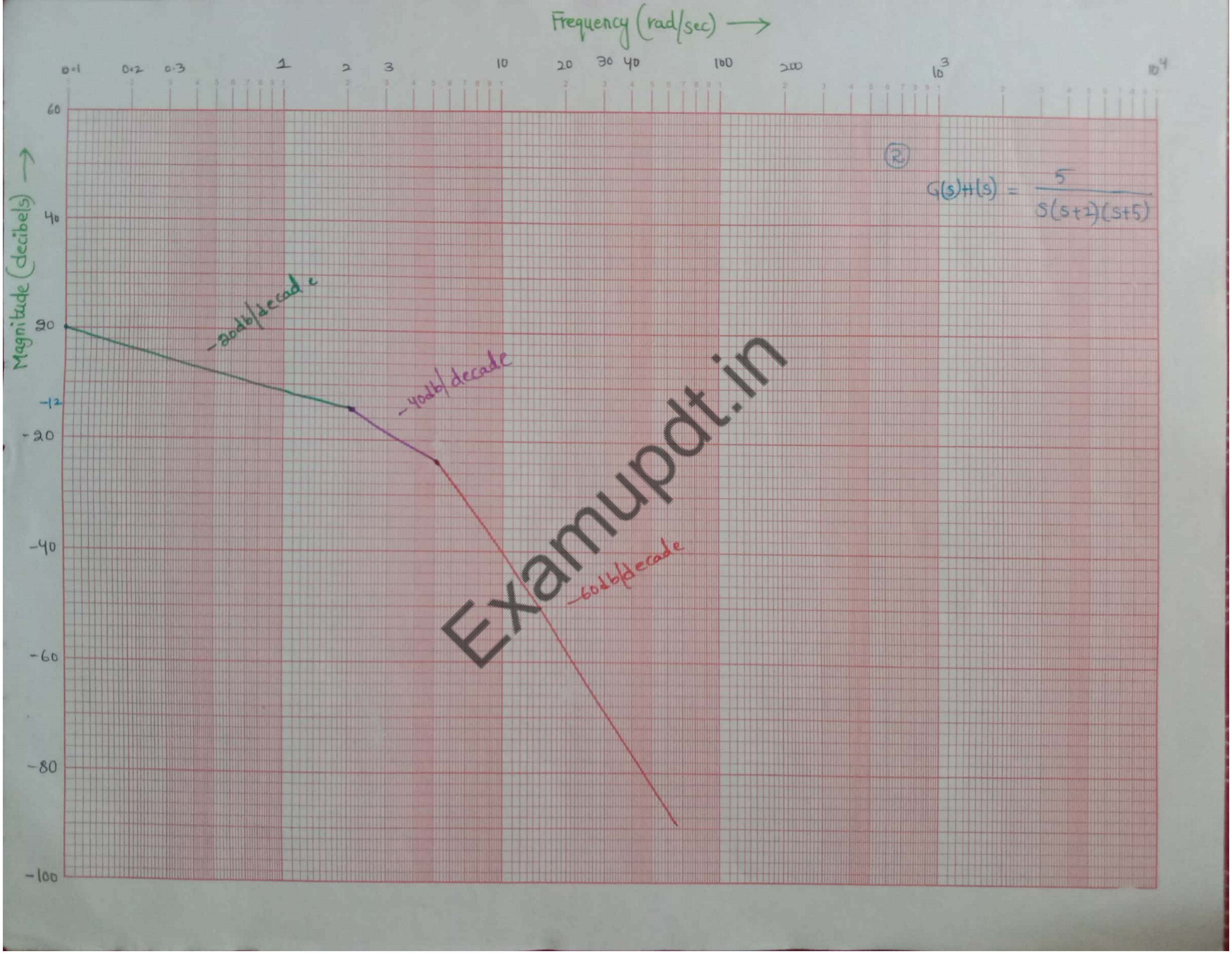
$$\angle G(j\omega)H(j\omega) = \angle 0.5 - \angle j\omega - (\angle 1+0.5j\omega) - \angle(1+j0.2\omega)$$

As they are denominator they have $-Ve$ sign.

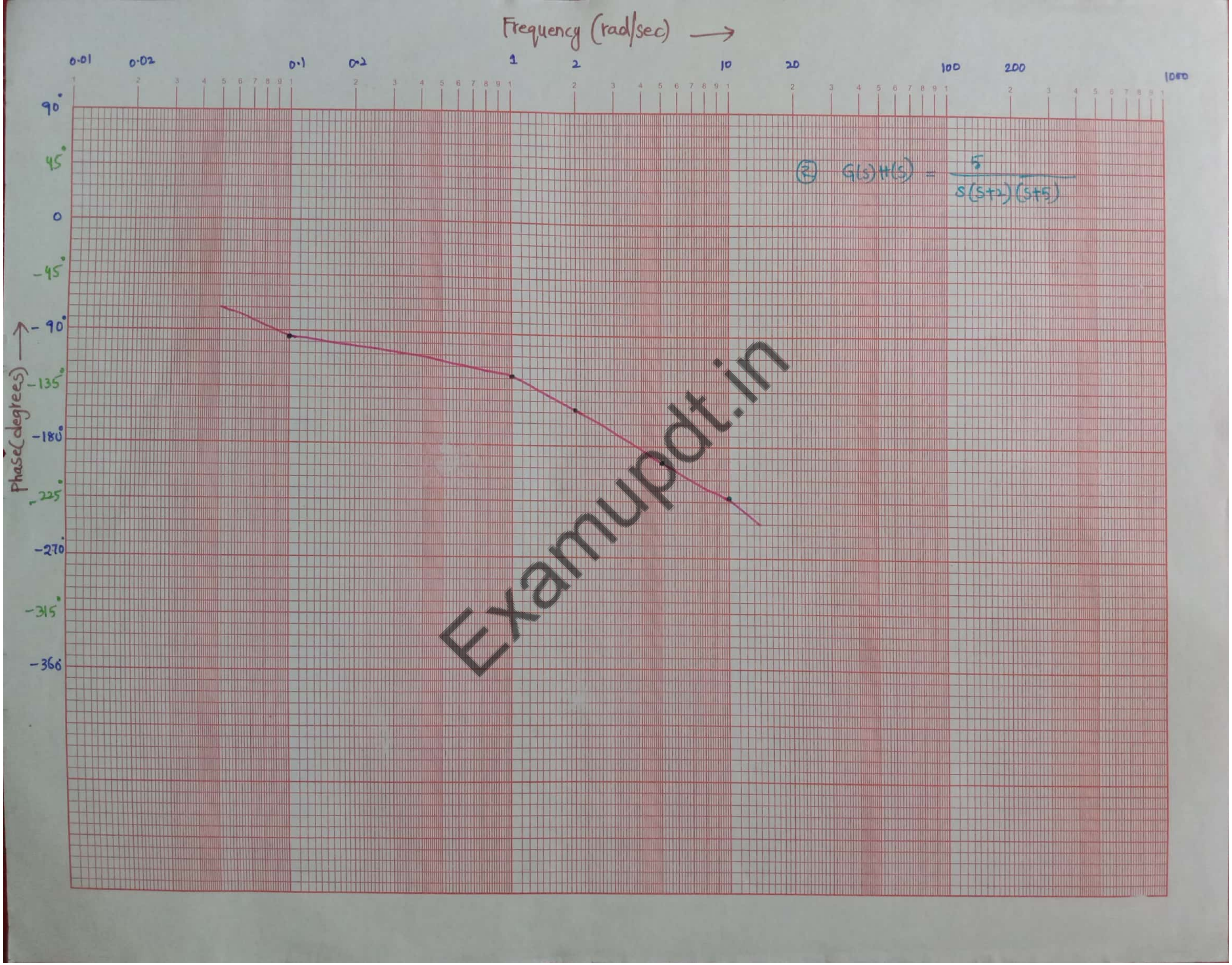
$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{0}{0.5}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{0.5\omega}{1}\right) - \tan^{-1}\left(\frac{0.2\omega}{1}\right)$$

$$\boxed{\angle j\omega = \tan^{-1}\left(\frac{\text{Imaginary}}{\text{Real}}\right)}$$

$$\boxed{\angle G(j\omega)H(j\omega) = 0 - 90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(0.2\omega)}$$



Frequency (rad/sec) →
0.1
0.2
0.3
1
2
3
10
20
30
40
100
200
10^3
10^4
Magnitude (decibels) →
60
40
20
-12
-20
-40
-60
-80
-100
②
G(s)H(s) = 5 / s(s+2)(s+5)
-20db/decade
-40db/decade
-60db/decade
Examupdt.in


Frequency (rad/sec)
0.01
0.02
0.1
0.2
1
2
10
20
100
200
1000
90°
45°
0
-45°
-90°
-135°
-180°
-225°
-270°
-315°
-360°
Phase (degrees)
② G(s)H(s) = 5 / s(s+2)(s+5)
Examupdt.in

ω	$\angle 0.5$ ①	$\angle j\omega$ ②	$\angle(1+j0.5\omega)$ ③ $\tan^{-1}(0.5\omega)$	$\angle(1+j0.2\omega)$ ④ $\tan^{-1}(0.2\omega)$	Total ①-②-③-④
0.1	0	$-90°$	-3	-1	-94
1	0	$-90°$	-26	-11	-127
2	0	$-90°$	-45	-22	-157
5	0	$-90°$	-68	-45	-203
10	0	$-90°$	-79	-63	-232

Example-1: (Another method

Draw the Bode plot of following Transfer function.

$$G(s) = \frac{20s}{s+10}$$

Sol:

Given,

$$G(s) = \frac{20s}{s+10}$$

$$= \frac{20s}{10(1+0.1s)}$$

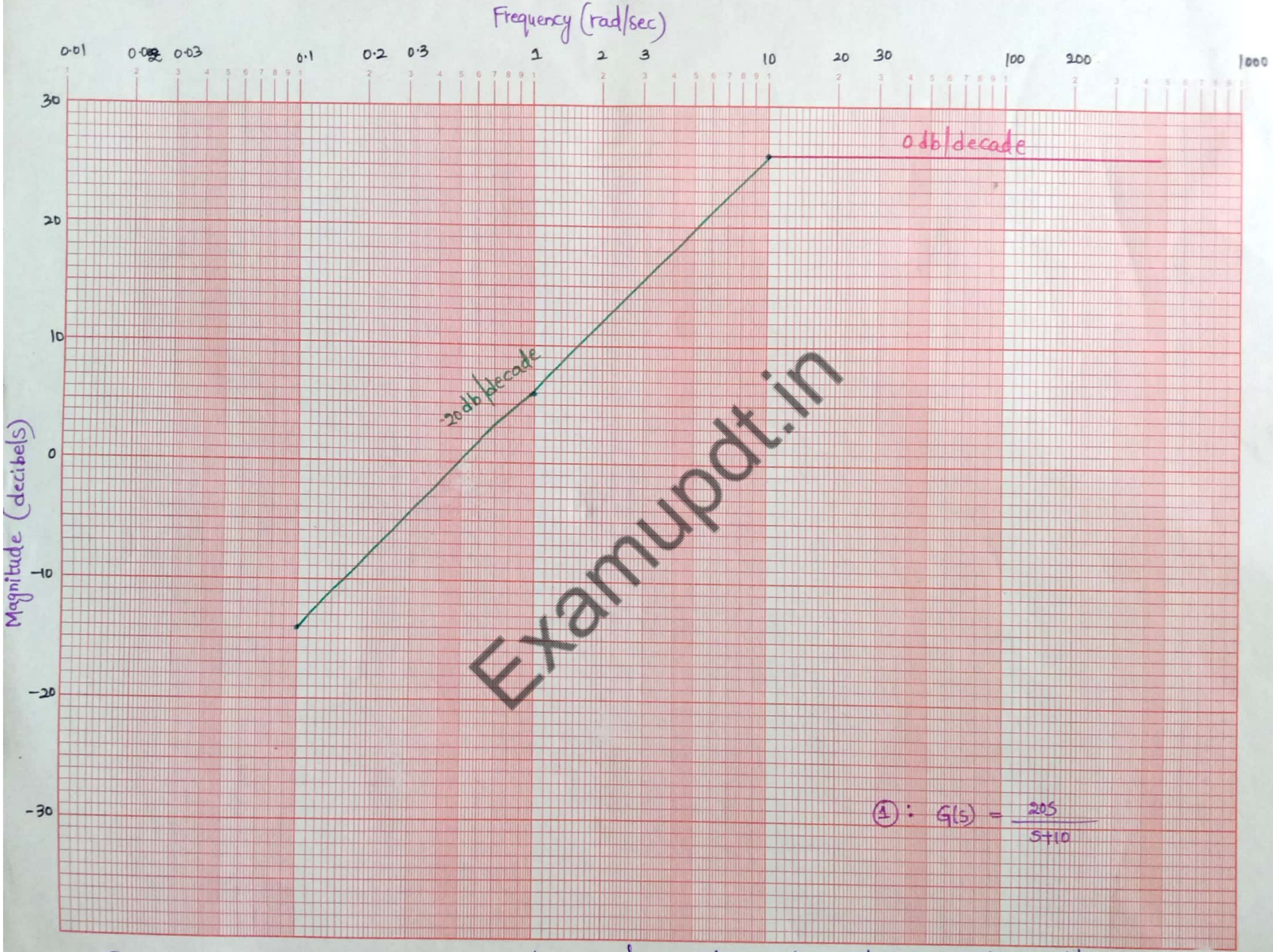
$$\boxed{G(s) = \frac{2s}{1+0.1s}}$$

The transfer function Contains:

1. Gain factor $(K=2)$
2. Derivative Factor (s)
3. 1st order Factor in denominator $(0.1s+1)^{-1}$

Magnitude plot:

Factor	Corner frequency(ω_c)	Slope	Magnitude
$2(j\omega)$	–	+20db/decade	$20\log(2\omega) = 20\log 2 + 20\log\omega$ $M = 20\log\omega + 6$ When $\omega = 1$, $M = 6dB$ Straight line of slope at +20db/decade low frequency Asymptote
$\frac{1}{(1+0.1\omega)}$	$\omega_{c_1} = 10$ rad/sec $T = 0.1$ $\omega = \frac{1}{T}$	-20 db/decade	Net slope: +20db/dec − 20db/dec Net slope = 0 db/dec



① : $G(s) = \frac{20s}{s+10}$

For this problem of another method we have same phase plot of this problem.

Here $\omega=1$ the GdB as it is 20db/decade. so

for 26db it will be $\omega=10$

-14db it will be $\omega=0.1$

Phase plot:

phase plot is same (No another method)

$$G(j\omega) = \frac{2j\omega}{1+j0.1\omega}$$

$$\angle G(j\omega) = \angle 2 + \angle j\omega - \angle (0.1j\omega+1)$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0}{2}\right) + \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{0.1\omega}{1}\right)$$

$$\boxed{\angle G(j\omega) = 90^\circ - \tan^{-1}(0.1\omega)}$$

ω	0.1	1	5	10	20	40	70	1000	∞
$\phi(\omega)$	89.4	84.2	63.4	45	26.5	14	8	0.5	0

At $\omega=10$ we have Corner Frequency. So the plotting will change after $\omega=10$ for magnitude plot.

Example-3:

$$G(s) = \frac{20(s+3)}{s(s+20)(s+100)}$$

Sol:

Given, $$G(s) = \frac{20(s+3)}{s(s+20)(s+100)}$$

$$= \frac{20(s+3)}{s(20(1+0.05s))\,100(1+0.01s)}$$

$$\boxed{G(s) = \frac{0.03(0.33s+1)}{s(1+0.05s)(1+0.01s)}}$$

Basic Factors:

1. Factor of form $\frac{K}{j\omega} = \frac{0.03}{j\omega}$

2. 1st order factor $(s+3)$

$$s+3 \Rightarrow (0.33s+1)$$

$$\therefore \boxed{\omega_{c_1} = 3}$$

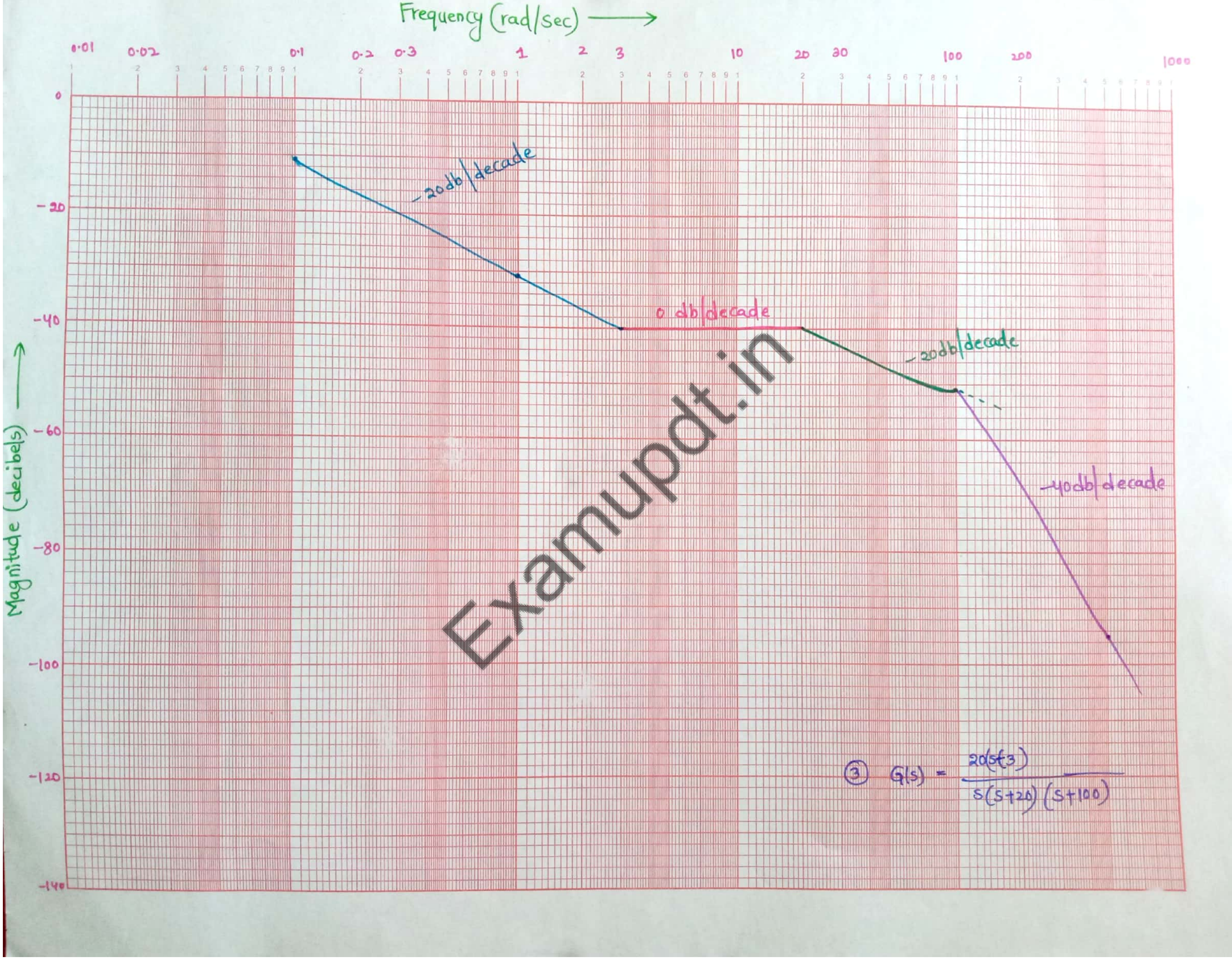
3. 1st order factor $(s+20)^{-1} \Rightarrow (0.05s+1)^{-1}$

$$\therefore \boxed{\omega_{c_2} = 20}$$

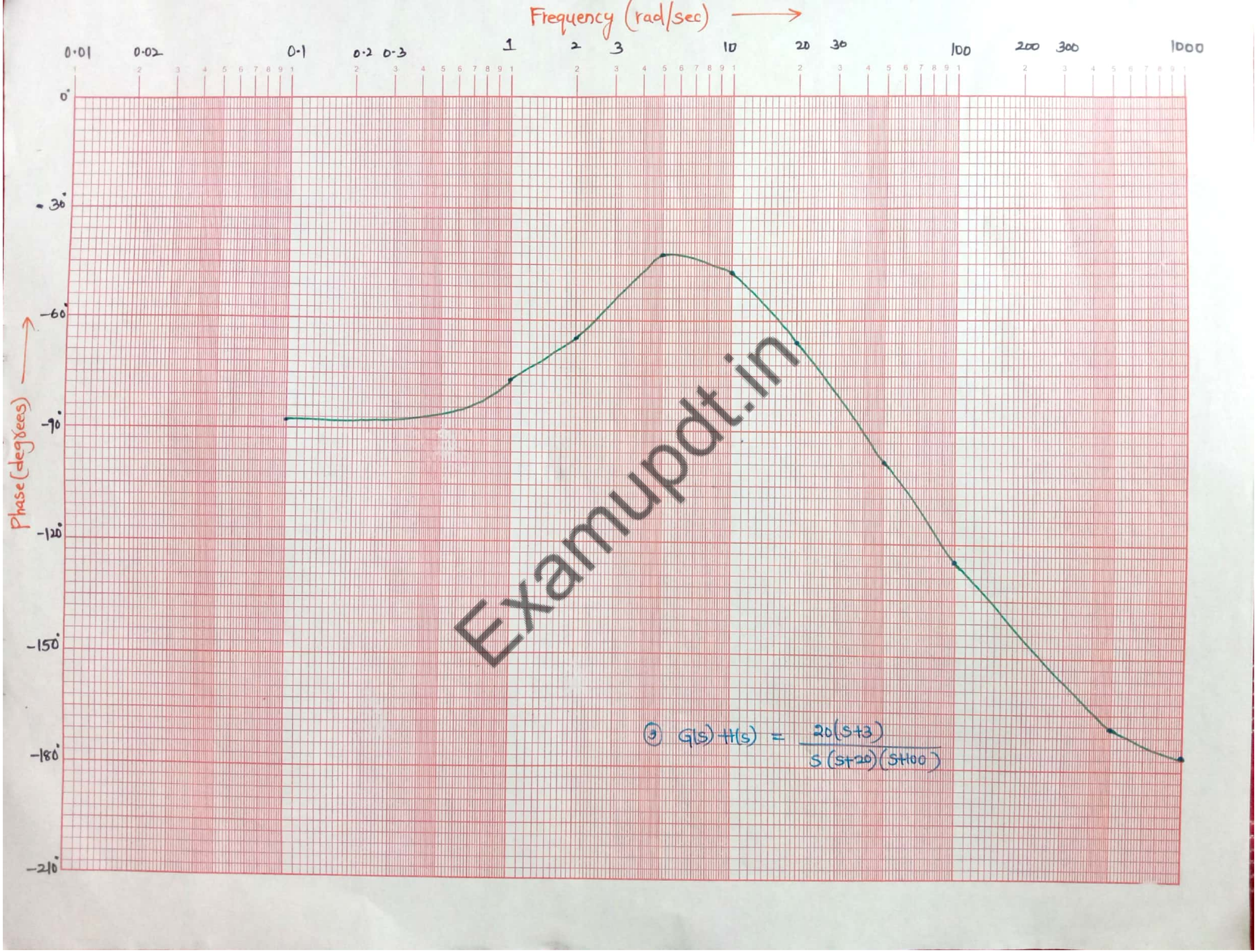
4. 1st order factor $\frac{1}{s+100} \Rightarrow (0.01s+1)^{-1}$

$$\therefore \boxed{\omega_{c_3} = 100}$$

Factor	Corner Frequency	Slope Contribution	Net Graph
$\frac{0.03}{j\omega}$	-	-20 db/dec $M = 20\log 0.03 - 20\log\omega$ When $\omega = 1$, $M = -30.45$	Straight line of slope -20db/dec passing through -30.45 db at $\omega = 1$ upto $\omega_{c_1} = 3$ rad/se Low Frequency Asymptote
$(1+j0.33\omega)$	$\omega_{c_1} = 3$ rad/sec	$W < 3$, $M = 0$ $W > 3$ straight line of slope $+20$ db/decade	For $\omega > 3$, net slope is 0 db/decade $(-20\text{db} + 20\text{db})$ Upto $\omega_{c_2} = 20$ rad/sec
$\frac{1}{1+j0.05s}$	$\omega_{c_2} = 20$ rad/sec	$\omega < 20$, $M = 0$ $\omega > 20$. straight line of slope -20 db/decade	For $\omega > 20$, net slope is -20db/decade (0db/decade -20db/decade)
$\frac{1}{1+j0.01\omega}$	$\omega_{c_3} = 100$ rad/sec	$\omega < 100$, $M = 0$ $\omega > 100$ straight line of Slope -20db/decade	For $\omega > 100$, net slope is -40db/decade (-20db/decade -20db/dec) Highest Frequency Asymptote



Frequency (rad/sec)
0.01
0.02
0.1
0.2
0.3
1
2
3
10
20
30
100
200
1000
0
-20
-40
-60
-80
-100
-120
-140
Magnitude (decibels)
-20db/decade
0 db/decade
-20db/decade
-40db/decade
③ G(s) = 20(s+3) / s(s+20)(s+100)



Frequency (rad/sec)
0·01
0·02
0·1
0·2
0·3
1
2
3
10
20
30
100
200
300
1000
Phase (degrees)
0°
-30°
-60°
-90°
-120°
-150°
-180°
-210°
Examupdt.in
(3) G(s) H(s) = $\frac{20(s+3)}{s(s+20)(s+100)}$

Phase plot :

$$G(j\omega)H(j\omega) = \frac{0.03(1+j0.33\omega)}{j\omega(1+j0.05\omega)(1+j0.01\omega)}$$

$$\phi = \tan^{-1}\left(\frac{0}{0.03}\right) + \tan^{-1}\left(\frac{0.33\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{0.05\omega}{1}\right) - \tan^{-1}\left(\frac{0.01\omega}{1}\right)$$

$$\boxed{\phi = 0^\circ + \tan^{-1}(0.33\omega) - 90^\circ - \tan^{-1}(0.05\omega) - \tan^{-1}(0.01\omega)}$$

ω	$\tan^{-1}(0.33\omega)$	-90°	$-\tan^{-1}(0.55\omega)$	$-\tan^{-1}(0.01\omega)$	Total
0.1	1.89	-90°	-0.286	-0.057	-88.453
1	18.26	-90°	-2.86	-0.5729	-75.1753
2	33.425	-90°	-5.710	-1.147	-63.432
5	58.78	-90°	-14.036	-2.86	-41.11
10	73.14	-90°	-26.5	-5.7	-49.06
20	81.38	-90°	-45	-11.31	-64.925
50	86.53	-90°	-68.19	-26.5	-98.16
100	88.26	-90°	-78.69	-45	-125.42
500	89.65	-90°	-87.70	-78.69	-166.74
1000	89.9	-90°	-88.42	-84.289	-172.886

4. Quadratic Factors $\left[1 + 2\xi(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]$

$$\boxed{M(\omega) = 20\log\sqrt{\underbrace{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}_{\text{Real}} + \underbrace{\left(2\xi\frac{\omega}{\omega_n}\right)^2}_{\text{Imaginary}}}}$$

$\because \; s^2 + 2\xi\omega_n s + \omega_n^2$

$$\omega_n^2\left[1 + \frac{2\xi j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2}\right]$$

$$\omega_n^2\left[1 + j\,2\xi\left(\frac{\omega}{\omega_n}\right) + j^2\left(\frac{\omega}{\omega_n}\right)^2\right]$$

For low frequencies $\omega << \omega_n$

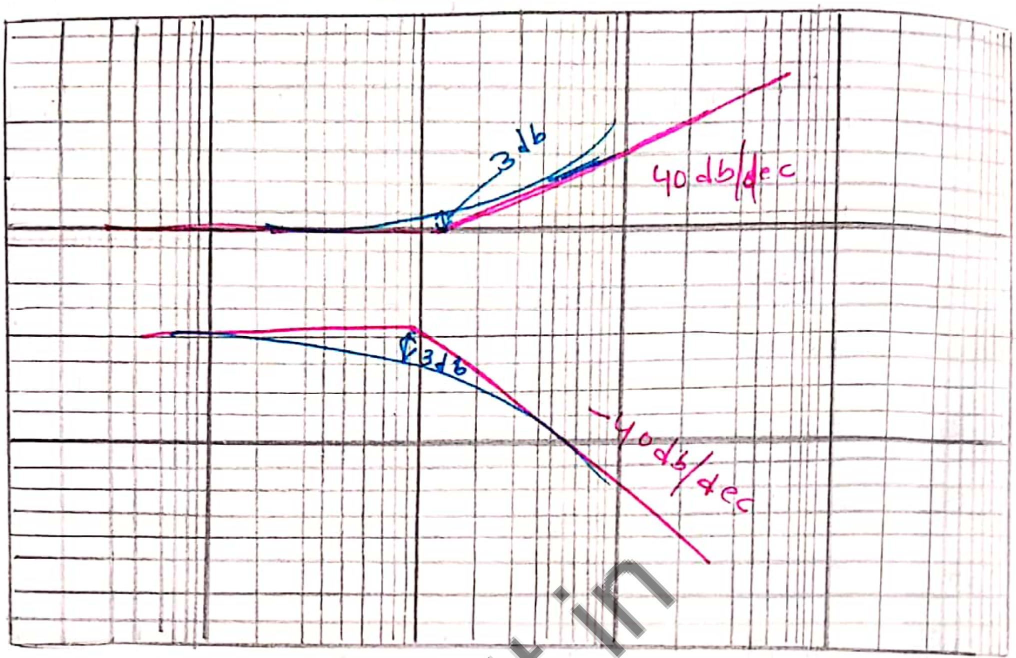
$$M(\omega) = 20\log(1)$$

$$\boxed{M(\omega) = 0}$$

For high frequencies $\omega >> \omega_n$

$$M(\omega) = 40\log\left(\frac{\omega}{\omega_n}\right)$$

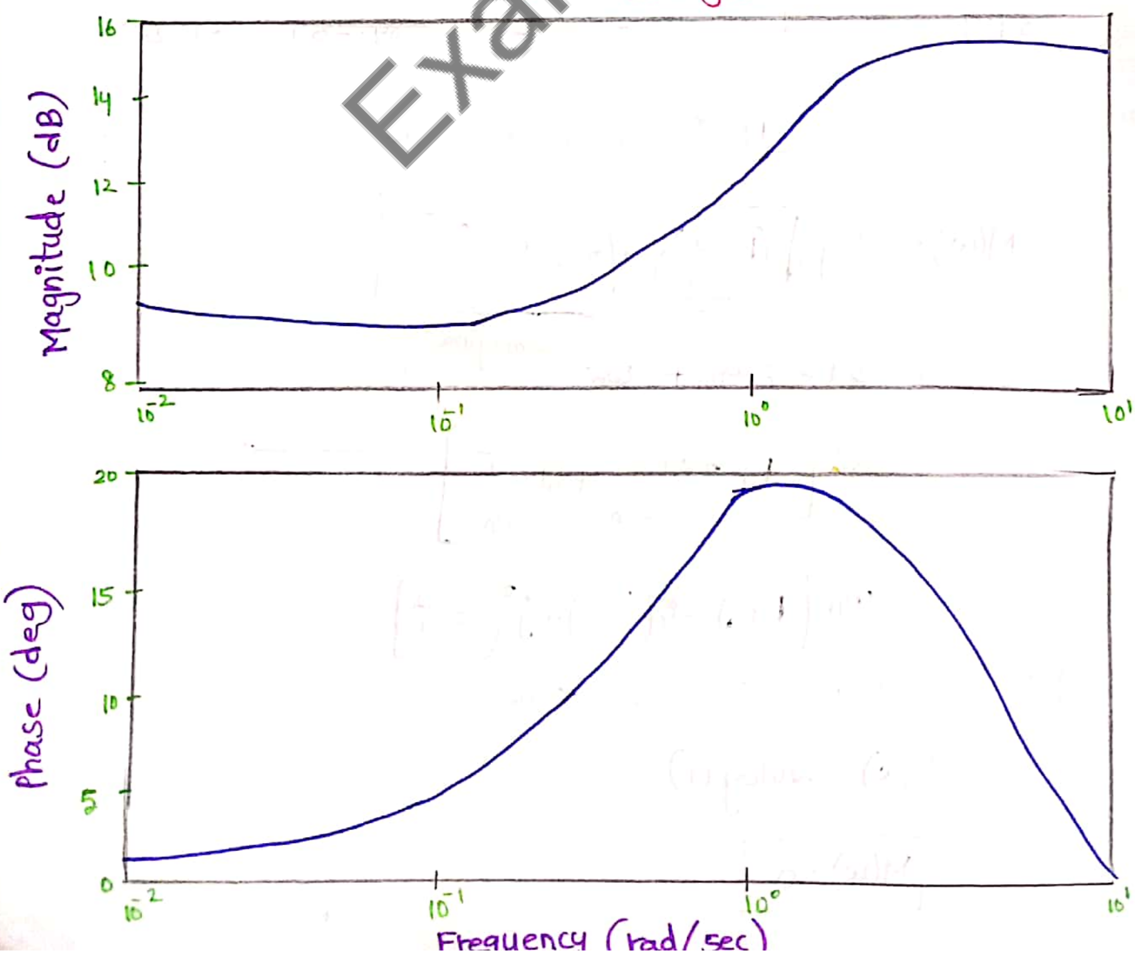
$$\boxed{M(\omega) = 40 \text{ db/decade}}$$



Minimum - Phase and Non-minimum phase Systems :

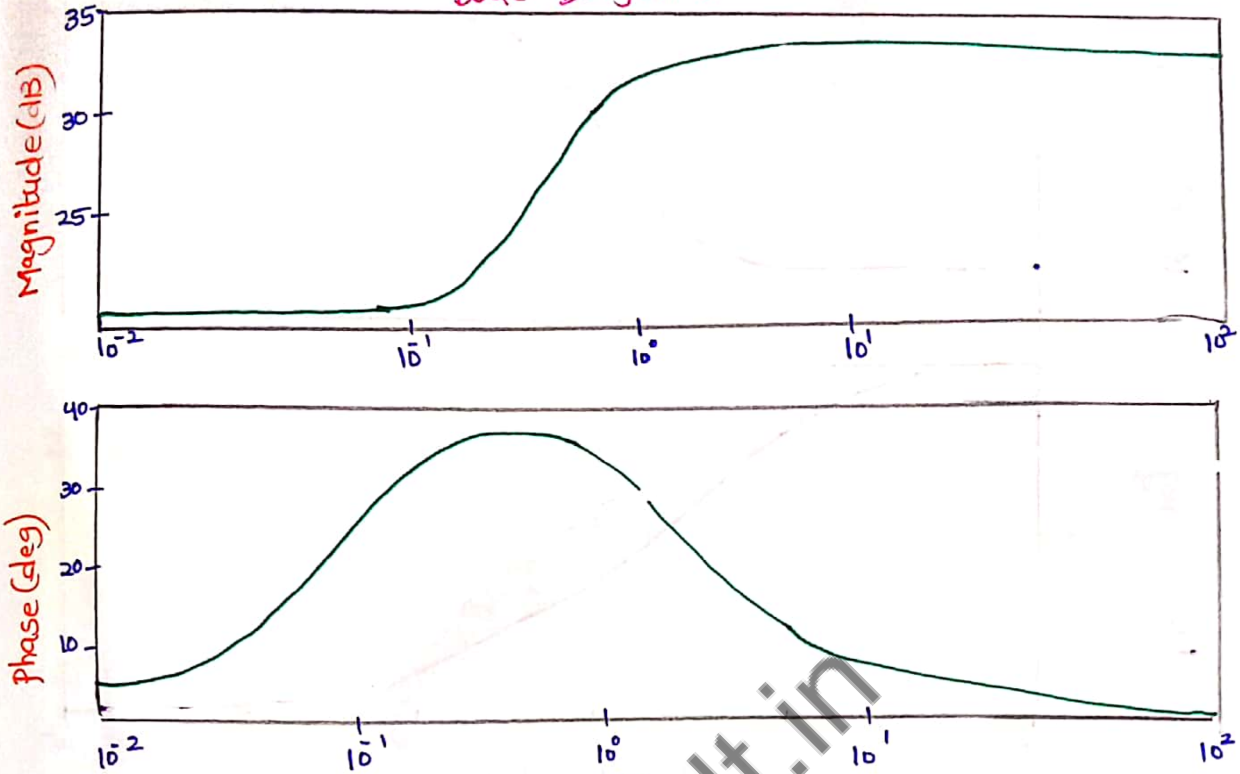
1) $G(s) = \dfrac{3(2s+1)}{(s+1)}$

Bode Diagram



2) $G(s) = \dfrac{10(3s+1)(7s+1)}{(s+1)(5s+1)}$

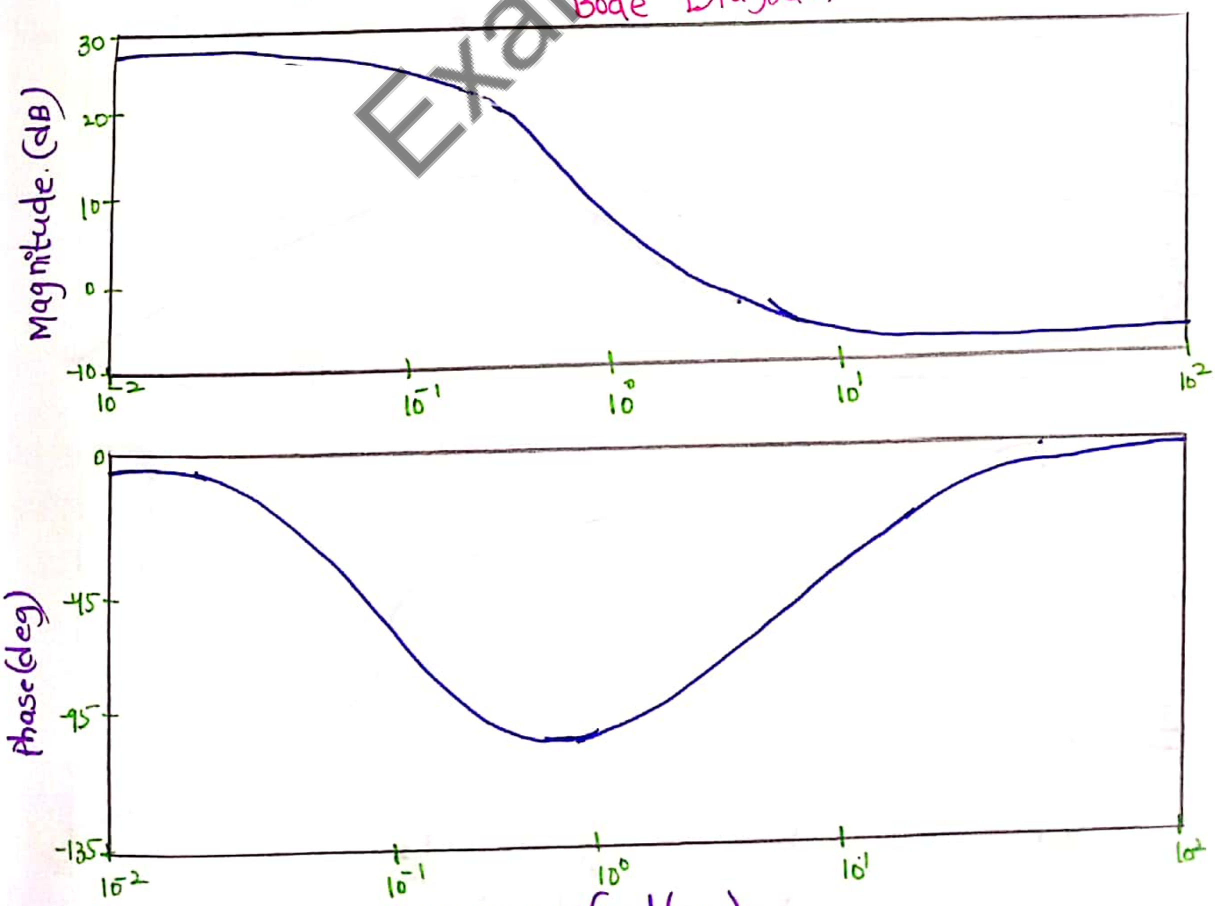
Bode Diagram



Frequency (rad/sec)

3) $G(s) = \dfrac{6(s^2+4s+4)}{(3s+1)(5s+1)}$

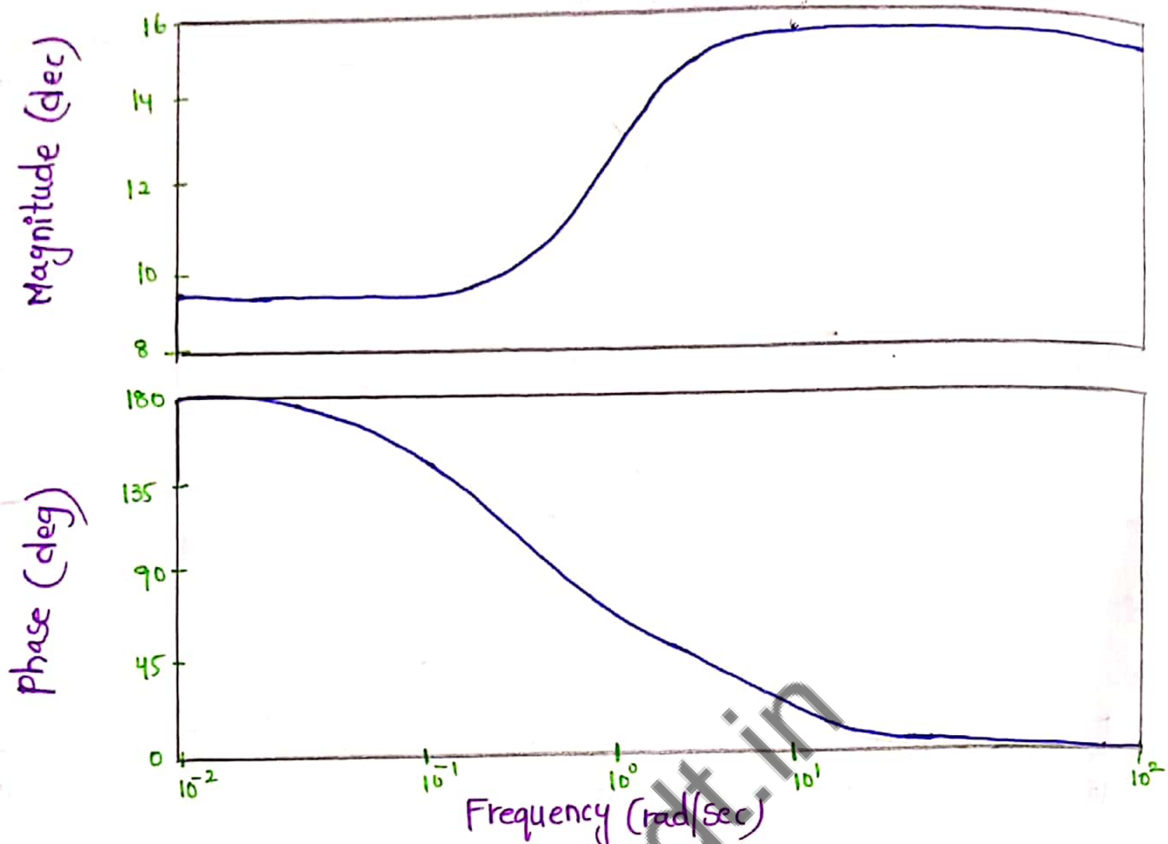
Bode Diagram



Frequency (rad/sec)

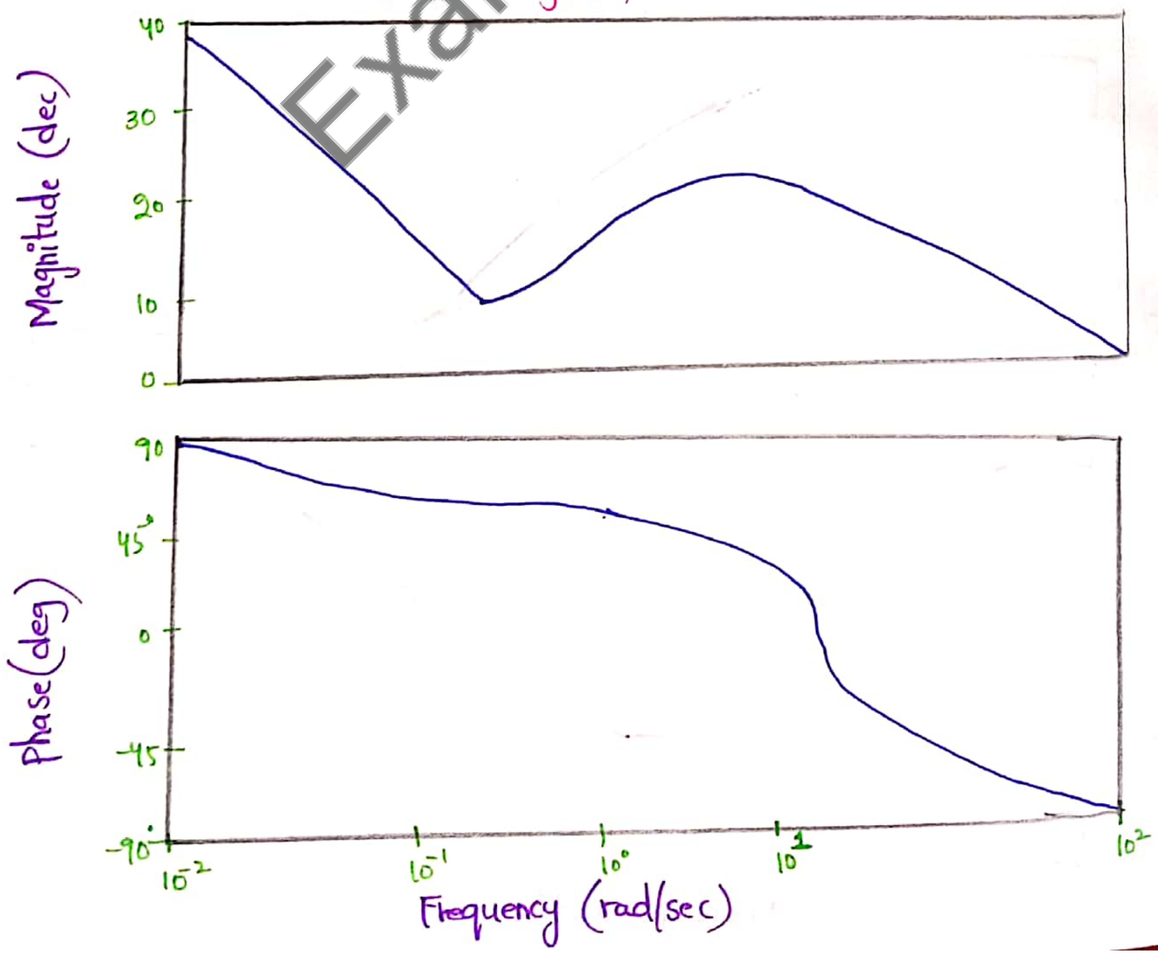
4) $G(s) = \dfrac{3(2s-1)}{(s+1)}$

Bode Diagram



5) $G(s) = \dfrac{10(5s-1)(2s+1)}{s(s^2+8s+16)}$

Bode Diagram



- Transfer functions having neither poles (nor) zeros in the right-half s-plane are minimum-phase transfer functions.
- Whereas, those having poles and/or zeros in the right-half s-plane are non-minimum-phase transfer functions.

Relative Stability:

1) **Phase crossover frequency (ω_p):**

It is the frequency at which the phase angle of the open-loop transfer function equals to $-180°$

2) **The gain crossover frequency (ω_g):**

It is the frequency at which the magnitude of the open loop transfer function, is unity.

3) **Gain Margin (K_g):**

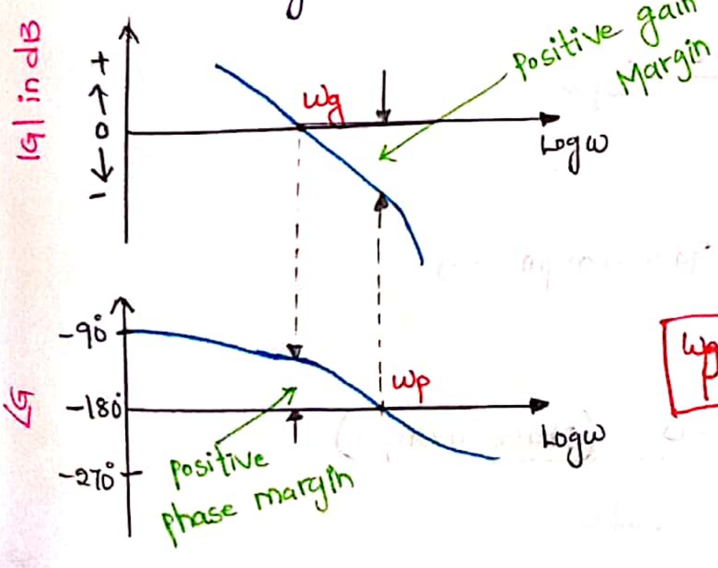
It is the reciprocal of the magnitude of $G(j\omega)$ at the Phase cross over frequency.

$$K_g = \frac{1}{|G(j\omega_1)|}$$

$$\boxed{K_g dB = 20 \log K_g = -20 \log |G(j\omega_1)|}$$

4) **Phase Margin (γ):**

It is the amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.



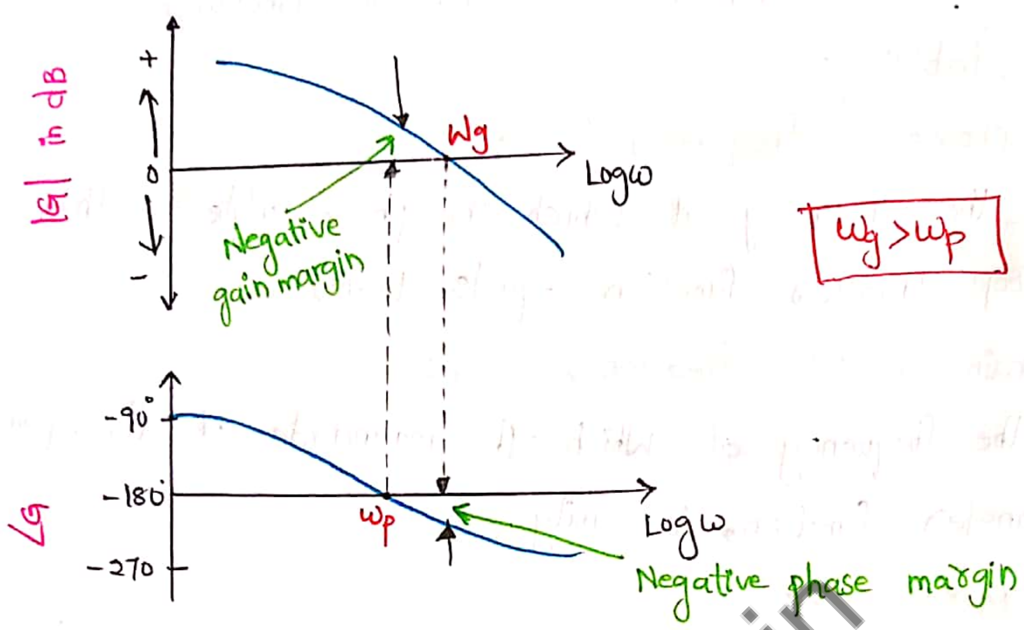
Stable system

$$\boxed{\omega_p > \omega_g}$$

For stable system,

phase margin $= \angle G(\omega_g) + 180^\circ \longrightarrow +ve$

Gain margin $= -\left(20\log\left(G(j\omega)\right)\right) \longrightarrow +ve$



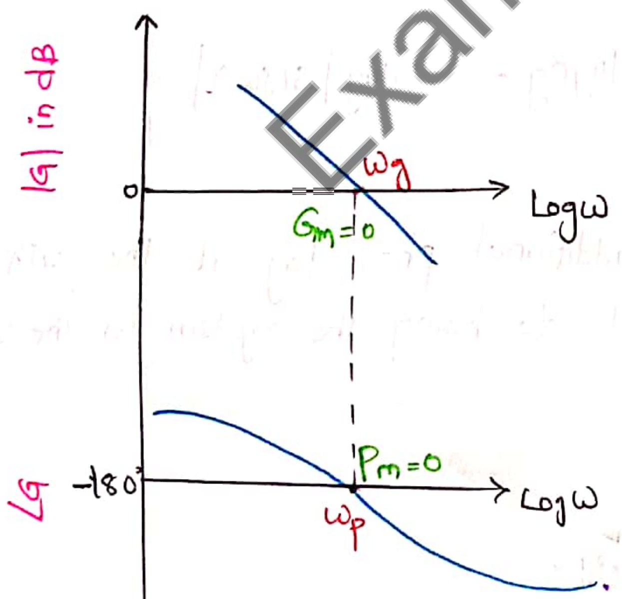
$\boxed{\omega_g > \omega_p}$

Unstable system

For Unstable system,

Phase margin = -ve

Gain margin = -ve



$\boxed{\omega_g = \omega_p}$

$\omega_p = \omega_g$ so Gain margin $= 0$

$\phi = -180^\circ$

$PM = -180 + 180^\circ = 0$ (Phase margin)

Marginally stable

For stable,

Phase margin = +ve

Gain margin = +ve

For marginally stable,

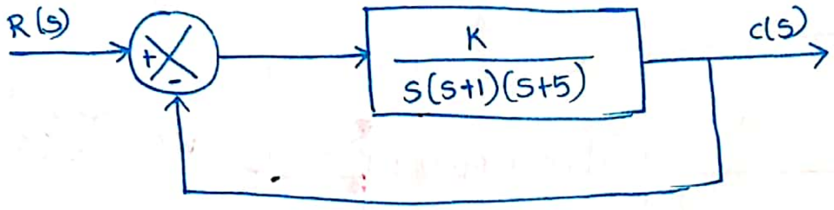
Gain margin = 0

Phase margin = 0

Remaining all other cases are Unstable.

Example-1:

Obtain the phase and gain margins of the system shown in following figure for the two cases where K=10 and K=100



Sol:

Given,

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

K=10:

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+5)}$$

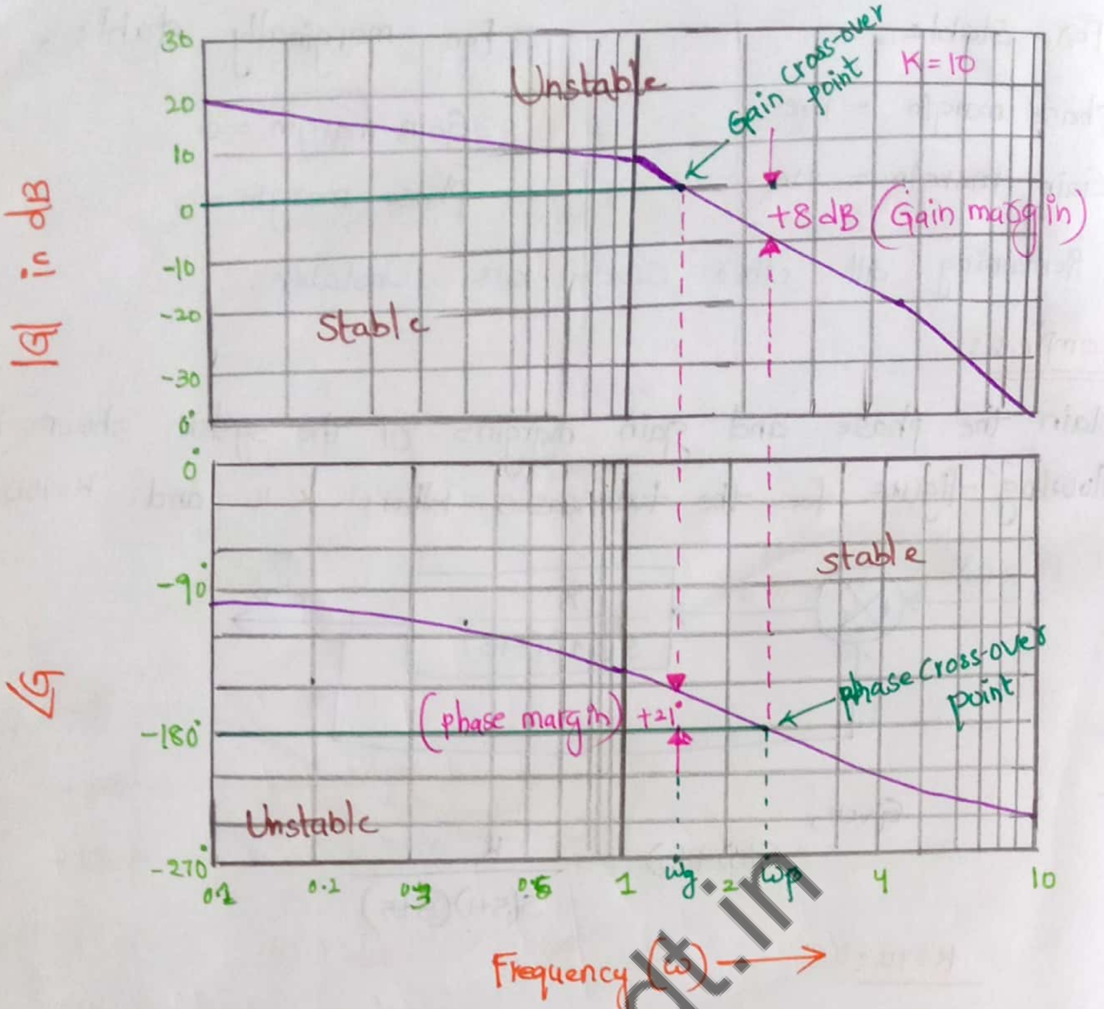
$$= \frac{10}{(j\omega)(1+j\omega)(5(1+0.2j\omega))}$$

$$\boxed{G(j\omega)H(j\omega) = \frac{2}{j\omega(1+j\omega)(1+j0.2\omega)}}$$

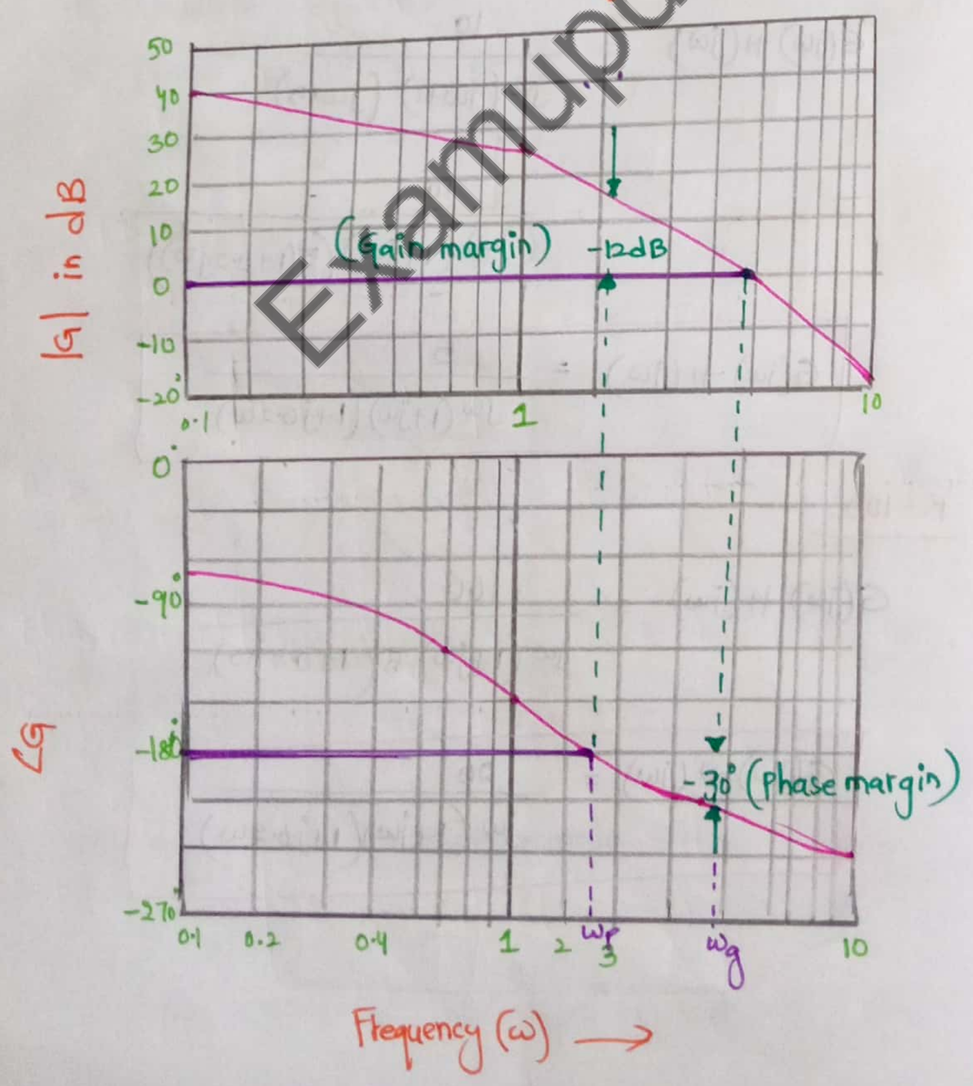
K=100:

$$G(j\omega)H(j\omega) = \frac{100}{j\omega(1+j\omega)5(1+0.2j\omega)}$$

$$\boxed{G(j\omega)H(j\omega) = \frac{20}{j\omega(1+j\omega)(1+j0.2\omega)}}$$



|G| in dB
30
20
10
0
-10
-20
-30
Unstable
Stable
Gain cross-over point
K=10
+8 dB (Gain margin)
∠G
0°
-90°
-180°
-270°
stable
Unstable
(phase margin) +21°
phase cross-over point
0.1
0.2
0.4
0.6
1
ωg
2
ωp
4
10
Frequency (ω)



|G| in dB
50
40
30
20
10
0
-10
-20
(Gain margin) -12dB
0.1
1
10
∠G
0°
-90°
-180°
-270°
-30° (Phase margin)
0.1
0.2
0.4
1
2
ωp
3
ωg
10
Frequency (ω)

Example-2:

Obtain Bode plot for $G(s) = \frac{100}{s(s^2+12s+100)}$

Sol: Given, $G(s) = \frac{100}{s(s^2+12s+100)}$

$$G(j\omega) = \frac{100}{j\omega\left(100\left[\frac{(j\omega)^2}{100} + \frac{12(j\omega)}{100} + 1\right]\right)}$$

$$\boxed{G(j\omega) = \frac{1}{j\omega\left[1 + 1.2\frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2\right]}}$$

2-factors:

1. Pole at origin $\frac{1}{s}$ (or) $\frac{1}{j\omega}$
2. Second order term in denominator.

Quad pole : $\frac{1}{1+1.2\left(\frac{j\omega}{10}\right)+\left(\frac{j\omega}{10}\right)^2}$

$\boxed{\omega_n = 10}$

$20\log K - 20\log\omega$

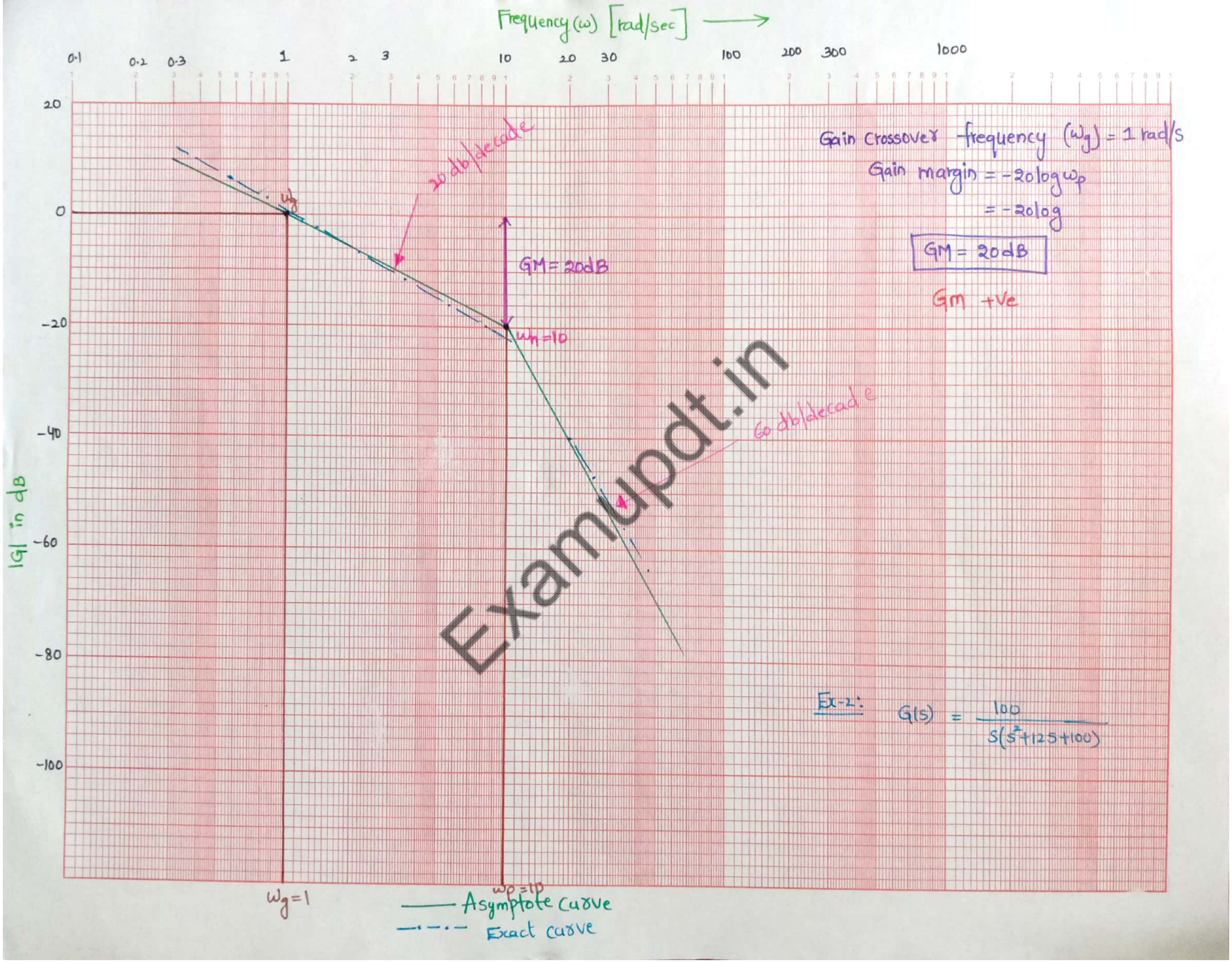
Factor	Corner Frequency	slope	Net slope	Remarks
$\frac{1}{j\omega}$	—	−20db/decade	−20 db/decade	straight line of slope −20db/decade passing (1,0)
Second order factor	$\omega_c = \omega_n = 10$	−40db/decade	−20db/decade −40db/decade = −60db/decade	Straight line of slope −60db/decade

Phase angle, $\phi = -\angle\frac{1}{s} - \angle$Quadratic equation

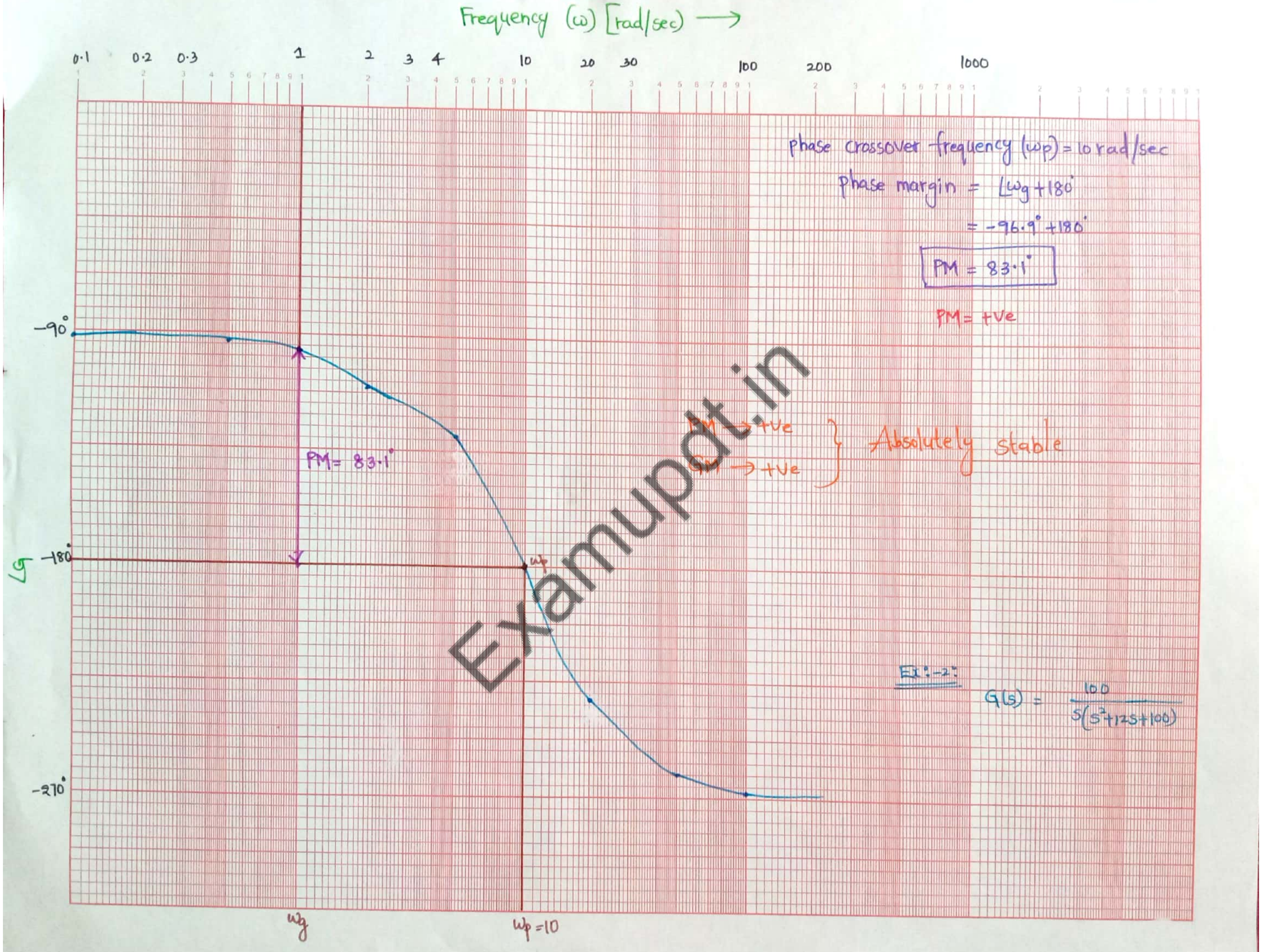
$$\boxed{\phi = -90^\circ - \tan^{-1}\left(\frac{0.12\omega}{1-0.01\omega^2}\right)}$$

$$\because \left[1 + 1.2\left(\frac{j\omega}{10}\right) + \left(\frac{j\omega}{10}\right)^2\right]^{-1}$$

$$\left[(1-0.01\omega^2) + j0.12\omega\right]^{-1}$$



Ex-2: $G(s) = \dfrac{100}{s(s^2+12s+100)}$


Frequency (ω) [rad/sec] →
0.1
0.2
0.3
1
2
3
4
10
20
30
100
200
1000
-90°
∠G -180°
-270°
phase crossover frequency (ωp) = 10 rad/sec
phase margin = ∠ωg + 180°
= -96.9° + 180°
PM = 83.1°
PM = +ve
PM = 83.1°
ωp
PM → +ve
GM → +ve
Absolutely stable
Ex:-2:
G(s) = 100 / s(s²+12s+100)
ωg
ωp = 10

ω	ϕ
0.1	-90.69
0.5	-93.44
1	-96.9
2	-104.03
5	-128.65
10	-180
20	
50	
100	

Gain cross-over frequency,

$$\omega_g = 1 \text{ rad/sec.}$$

phase cross-over frequency,

$$\omega_p = 10 \text{ rad/sec} \quad (\text{at } -180^\circ)$$

Gain Margin,

$$GM = -20 \log(\omega_p)$$

$$\boxed{GM = +20\,dB}$$

Phase Margin,

$$PM = \angle\omega_g + 180^\circ = -96.9 + 180^\circ$$

$$\boxed{PM = 83.1^\circ}$$

Both are positive then the system is Absolutely stable

Mathematical Modelling to find Transfer function:

- Mathematical modelling is used to find Transfer function.
- Output for different frequencies
- If input and output are known then,

$$\boxed{Gain = 20 \log\left(\frac{C(s)}{R(s)}\right)}$$

ω	R(s)	C(s)	$\frac{C(s)}{R(s)}$	$20 \log \left[\frac{C(s)}{R(s)}\right]$

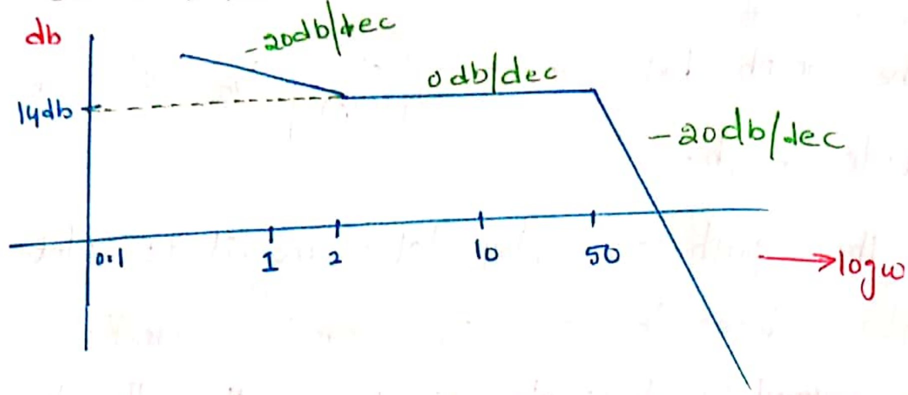
- Plot the graph between $20 \log\left[\frac{C(s)}{R(s)}\right]$ Vs 'ω' which is magnitude graph.
- Make the graph as Asymptotic magnitude plot.

Experimental determination of Transfer function

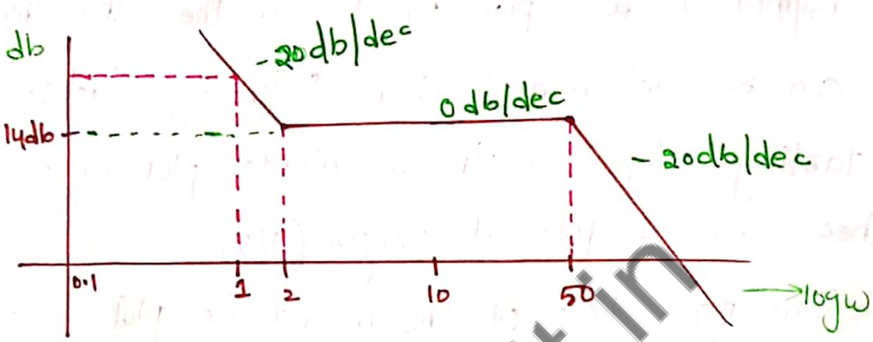
- If the asymptotic bode plot is known, then the transfer function can be determined from following facts.
- If the starting slope of the magnitude plot is −20db/decade, then there is 1 pole at origin $(1/s)$
- If the starting slope of the magnitude plot is −40 db/dec, then there are 2 poles at origin $(1/s^2)$
- If the starting slope of the magnitude plot is +20 db/dec, then there is a zero at origin (s).
- The shift in magnitude at $\omega=0$, gives us the gain $M = 20\log k$
- The frequency at which the slope is changing will give the corner frequency (ω_c). If the change is slope is −20db/dec, then there exists a pole $\left\{\frac{1}{1+Ts}\right\}$ and if the change in slope is +20db/decade. then there is zero $(1+Ts)$ Where $T = \frac{1}{\omega_c}$
- Second order factors have a change in slope of ±40db/dec for zero (or) pole.

Problem-1:

1) Determine the Transfer Function for the System with magnitude plot shown below



Sol:



① $\frac{1}{S}$

$-20db/dec$

$M = 20\log K - 20\log\omega$

At $\omega = 2$, $M = 14dB$

$14 = 20\log k - 20\log 2$

$20\log k = 14 + 20\log 2$

$= 14 + 6$

$20\log k = 20$

$\boxed{K = 10}$

② 0 db/dec

At $\omega = 2$, Slope changes from -20 db/decade to 0 db/dec

$\therefore \boxed{\text{slope} = +20db/dec}$

First order factor in numerator (zero)

$(1+j\omega T) = (1+j\omega \times \frac{1}{2}) = (1+j\omega(0.5)) = (1+0.5S)$

③ $-20\,db/dec$

At $\omega=50$, slope changes from $0\,db/dec$ to $-20\,db/decade$

$\therefore$ $\boxed{-20\,db/decade.}$

$\therefore$ First order factor in denominator (pole).

$$\frac{1}{(1+j\omega T)} = \frac{1}{\left(1+j\frac{\omega}{50}\right)} = \frac{1}{1+j0.02\omega} = \frac{1}{1+0.02S}$$

$$\therefore G(s) = k \times \frac{1}{s} \times (1+j\omega T) \times \frac{1}{(1+j\omega T)}$$

$$= \frac{10(1+0.5S)}{S(1+0.02S)}$$

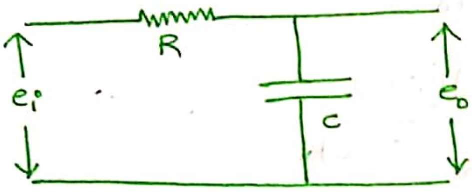
$$= \frac{10\left(\frac{1}{2}\right)(S+2)}{S \times \frac{1}{50}(S+50)}$$

$$\therefore \boxed{G(S) = \frac{250(S+2)}{S(S+50)}}$$

Polar plots:

- In bode plots, We plot the Magnitude Versus frequency and phase Angle Versus frequency. We have two different plots.
- Polar plot is a plot drawn between magnitude and phase angle as frequency Varies from 0 to ∞.
- Usually plotted in the polar graph sheet but can also be plotted in a normal graph paper by converting the polar coordinates into Cartesian Coordinates.
- Sometimes also called the Nyquist plot.
- The polar plot is usually plotted on the polar graph sheet. Polar graph sheet has Concentric circles and radial lines. The circles represent the magnitude and the radial lines represent the phase angles.

- The phase angle is measure positive in the Anti-clockwise direction with the reference axis.
- To plot the polar plot, the magnitude $|G(j\omega)|$ and the phase angle $\angle G(j\omega)$ are computed for various values of ω and tabulated. Usually the choice of frequencies are 0, corner frequencies and frequencies near the corner frequency and infinity.
- The polar plot can also be plotted on the regular graph by converting the polar magnitude and phase angle into complex values and plotting the same



$$e_i(t) = Ri(t) + \frac{1}{C}\int i(t)$$

$$\boxed{E_i(s) = \left[R + \frac{1}{cs}\right] I(s)}$$

$$e_o(t) = \frac{1}{c}\int i(t)$$

$$\boxed{E_o(s) = \frac{1}{cs} I(s)}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{cs}}{R + \frac{1}{cs}} = \frac{1}{Rcs + 1}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Rc}}{s + \frac{1}{Rc}}}$$

Let $T = Rc$

$$G(s) = \frac{E_o}{E_i} = \frac{\frac{1}{T}}{s + \frac{1}{T}} = \frac{1}{1+Ts}$$

Sinusoidal Transfer function $(s = j\omega)$

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

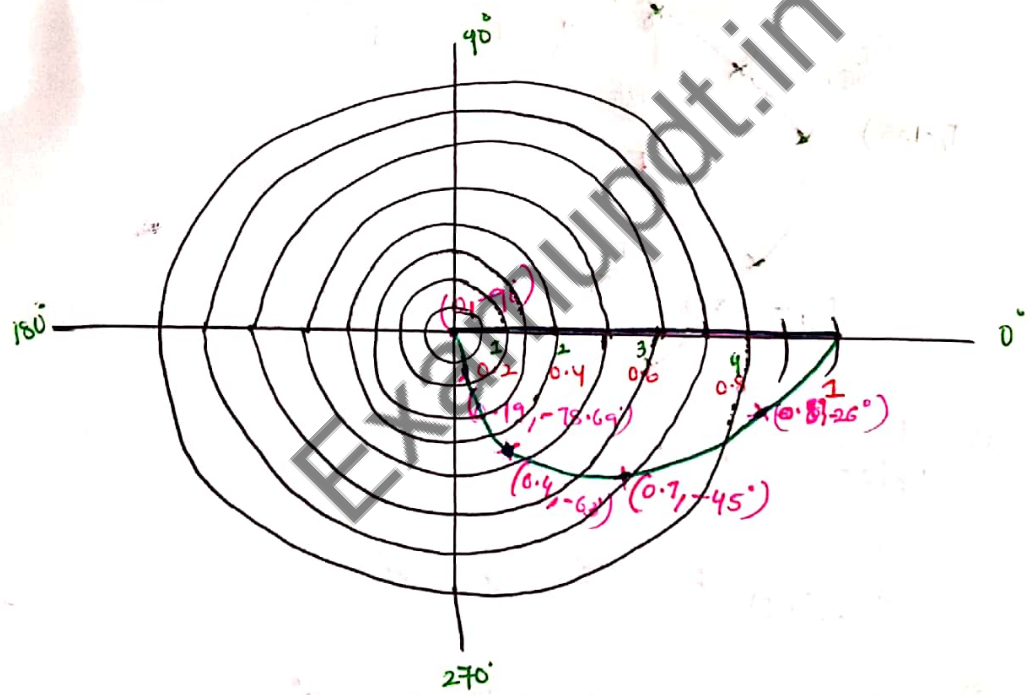
$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T^2}} \qquad \angle G j\omega = -\tan^{-1}(\omega T)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$Re\, G(j\omega)$	$Im\, G(j\omega)$
0	1	0	1	0
$\frac{1}{2T}$	$\sqrt{\frac{4}{5}}$	-26°	0.8039	-0.3524
$\frac{1}{T}$	$\frac{1}{\sqrt{2}}$	-45°	-0.3536	-0.5
$\frac{2}{T}$	$\frac{1}{\sqrt{5}}$	-63.4	0.2002	-0.179
$\frac{5}{T}$	$\frac{1}{\sqrt{26}}$	-78.69°	0.0385	-0.0377
∞	0	-90°	0	0

$Re(G(j\omega)) = Real\,[G(j\omega)]$

$Im[G(j\omega)] = Imaginary\,[G(j\omega)]$

(γ, θ) $\quad Re \longrightarrow \gamma\cos\theta \quad Im \longrightarrow \gamma\sin\theta$



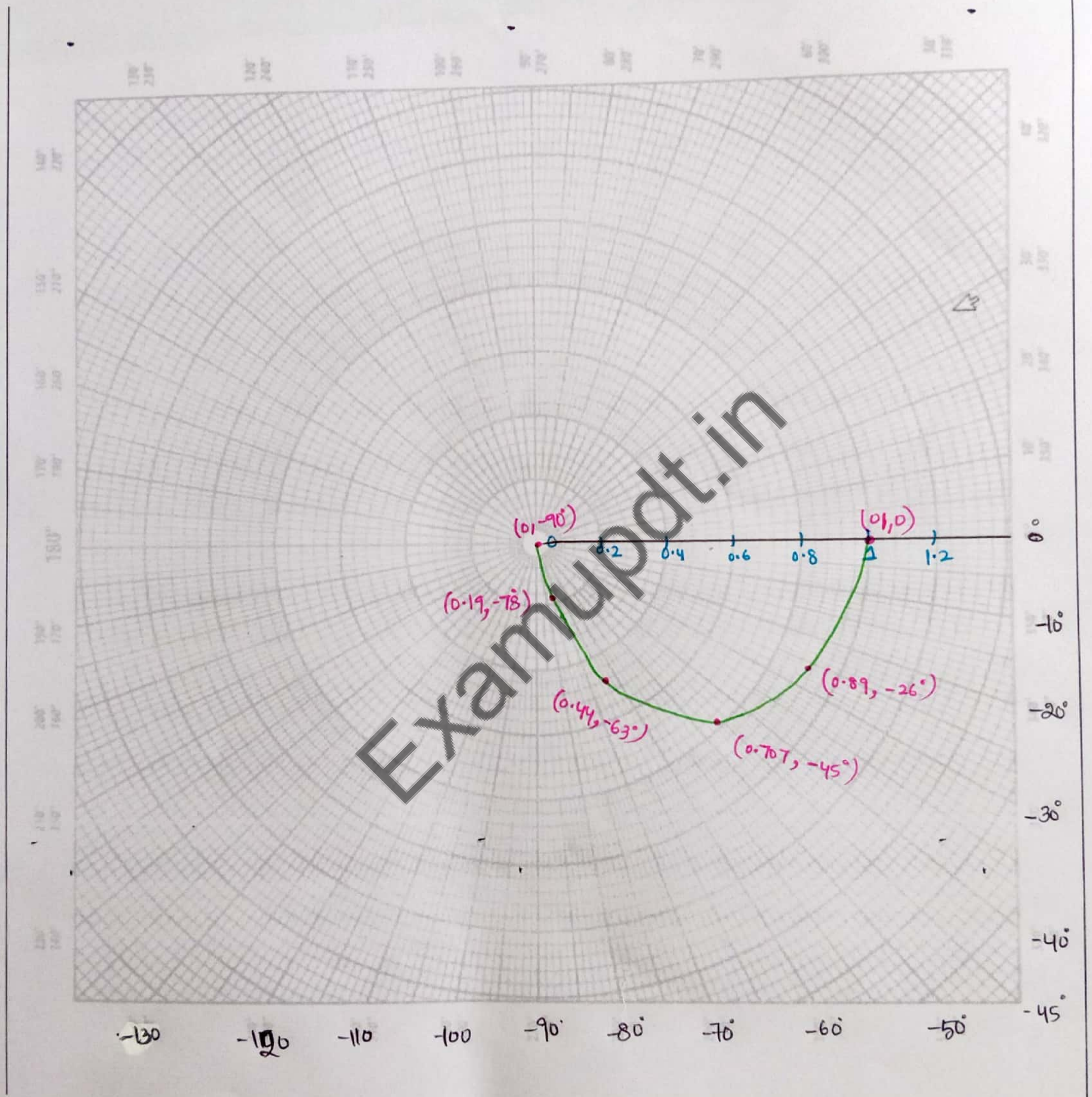
Example-I:

$$G(j\omega) = \frac{1}{(j\omega)(1+j\omega T)}$$

Sol: Given,

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

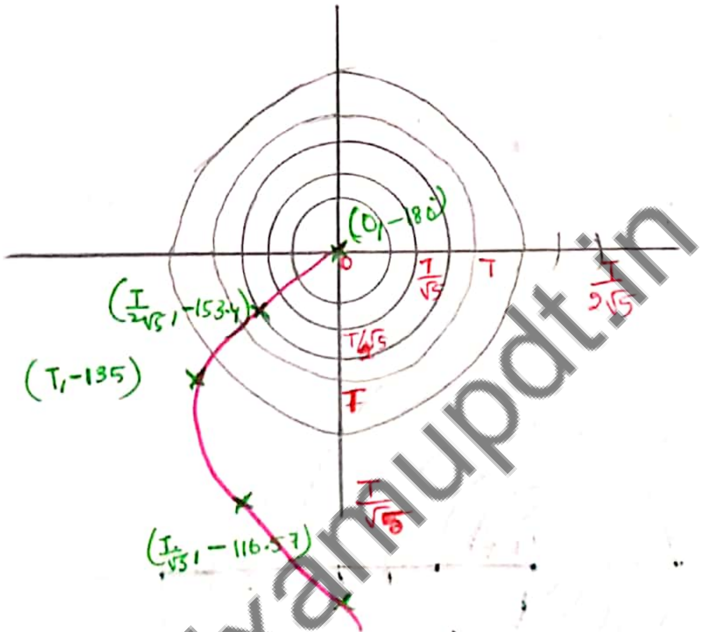
$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2T^2}}$$



$$G(j\omega) = \frac{1}{1+j\omega T}$$

$$\angle G(j\omega) = \phi = -90^\circ - \tan^{-1}(\omega T)$$

ω	M	ϕ
0	∞	-90°
$\frac{1}{2T}$	$\frac{T}{\sqrt{5}}$	-116.57
$\frac{1}{T}$	T	-135°
$\frac{2}{T}$	$\frac{T}{2\sqrt{5}}$	-153.43
∞	0	-180°



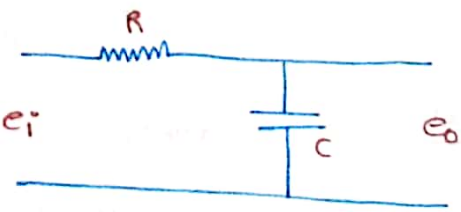
1) $\frac{1}{1+j\omega T}$: $0 \longrightarrow -90^\circ$

2) $\frac{1}{j\omega(1+j\omega T)}$: $-90^\circ \longrightarrow -180^\circ$

3) $\frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$: $-90^\circ \longrightarrow -270^\circ$

First order system

Time Constant of 0.1 s



$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{1}{1+RCs}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{1+sT}}$$

$T = 0.1S$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{1+0.1S}$$

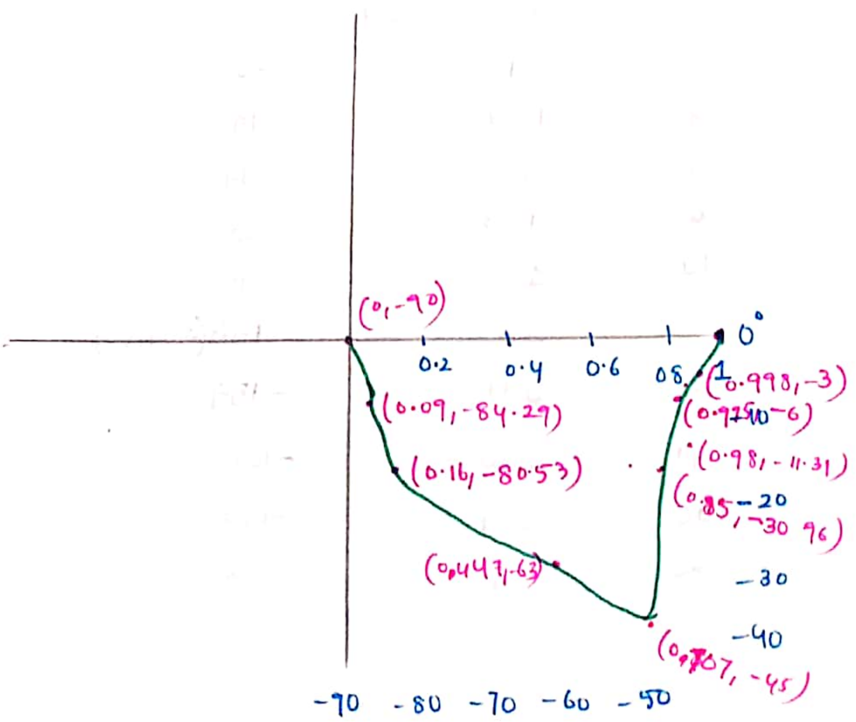
$$G(j\omega) = \frac{1}{1+j0.1\omega}$$

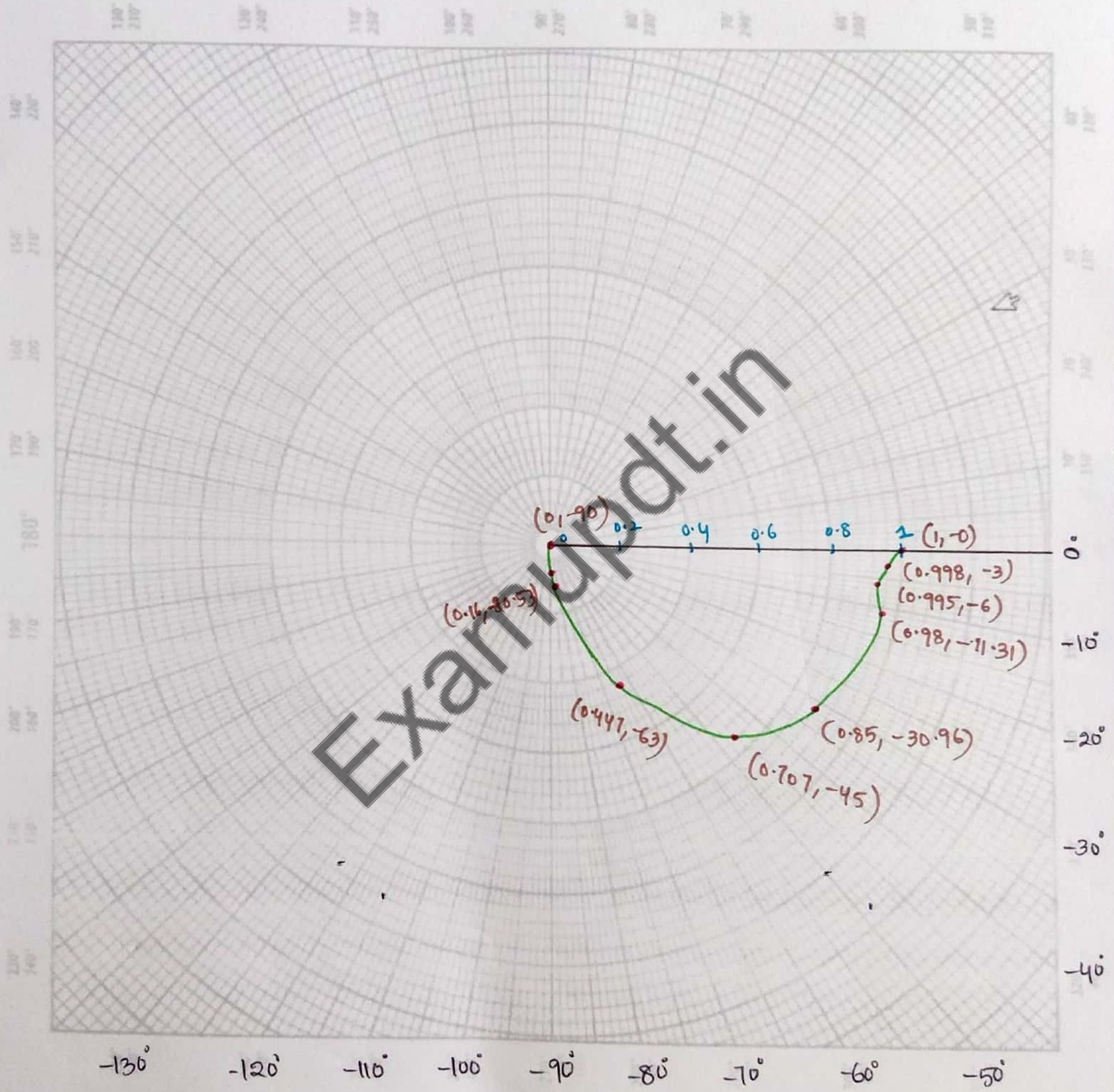
$$\boxed{M(j\omega) = \frac{1}{\sqrt{1+0.01\omega^2}}}$$

$$\phi = -\tan^{-1}(\omega T)$$

$$\boxed{\phi = -\tan^{-1}(0.1\omega)}$$

ω	$M(j\omega)$ $M = \frac{1}{\sqrt{1+0.01\omega^2}}$	$\phi(j\omega)$ $-\tan^{-1}(0.1\omega)$
0	1	-0
0.5	0.998	-3
1	0.995	-6
2	0.98	-11.31
6	0.85	-30.96
10	0.707	-45
20	0.447	-63
60	0.16	-80.53
100	0.09	-84.29
∞	0	-90





Time constant = 0.1s

$$G(j\omega) = \frac{1}{1+j0.1\omega}$$

Problem:

A second order system with $\omega_n = 10$ and $\xi = 0.5$. Draw the Polar plot

Sol:

$$\frac{C(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\boxed{G(s) = \frac{100}{s^2+10s+100}}$$

$$G(j\omega) = \frac{100}{(j\omega)^2+10(j\omega)+100}$$

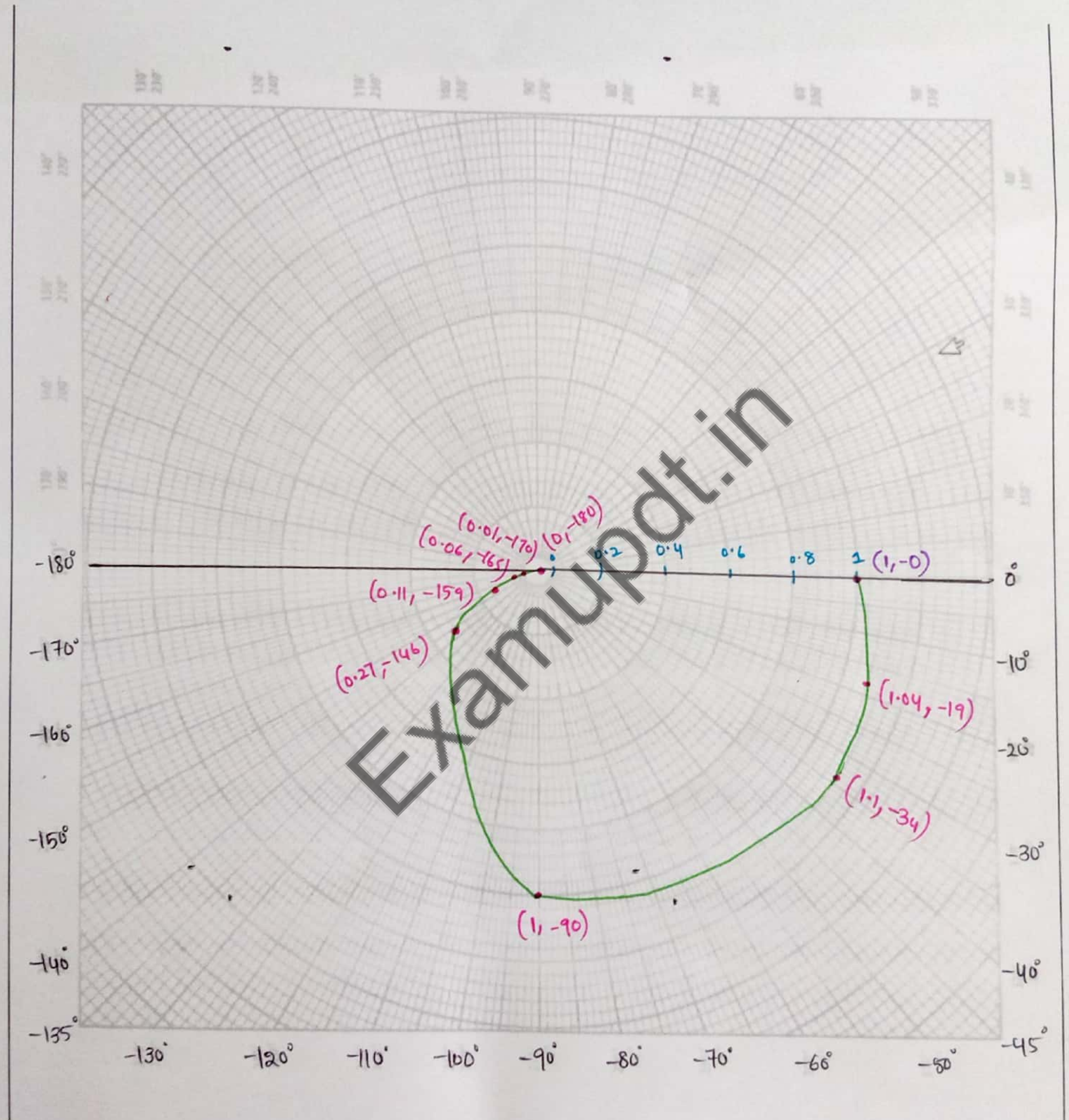
$$= \frac{100}{(100-\omega^2)+j(10\omega)}$$

$$\boxed{G(j\omega) = \frac{1}{[1-(0.1\omega)^2]+j(0.1\omega)}}$$

$$|M(j\omega)| = \frac{1}{\sqrt{(1-0.01\omega^2)^2+(0.01\omega^2)}}$$

$$\phi = -\tan^{-1}\left(\frac{0.1\omega}{1-0.01\omega^2}\right)$$

ω	$M = \frac{1}{\sqrt{[1-0.01(\omega^2)]^2+0.01\omega^2}}$	$\phi = -\tan^{-1}\left(\frac{0.1\omega}{1-0.01\omega^2}\right)$
0	1	−0
3	1.04	−19
5	1.1	−34
8	1.13	−65
10	1	−90
20	0.27	−146 (−34+180)
30	0.11	−159
40	0.06	−165
100	0.01	−170
∞	0	−180

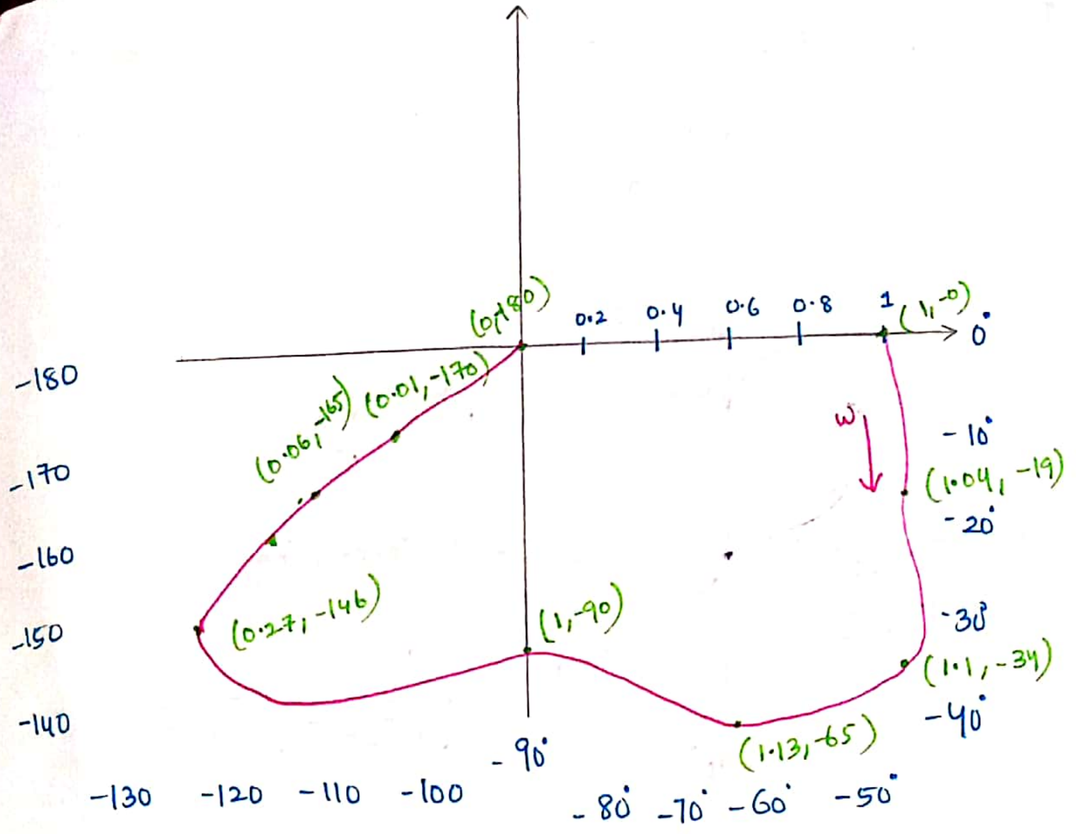


$$G(j\omega) = \frac{1}{[1-(0.1\omega)^2] + j(0.1\omega)}$$

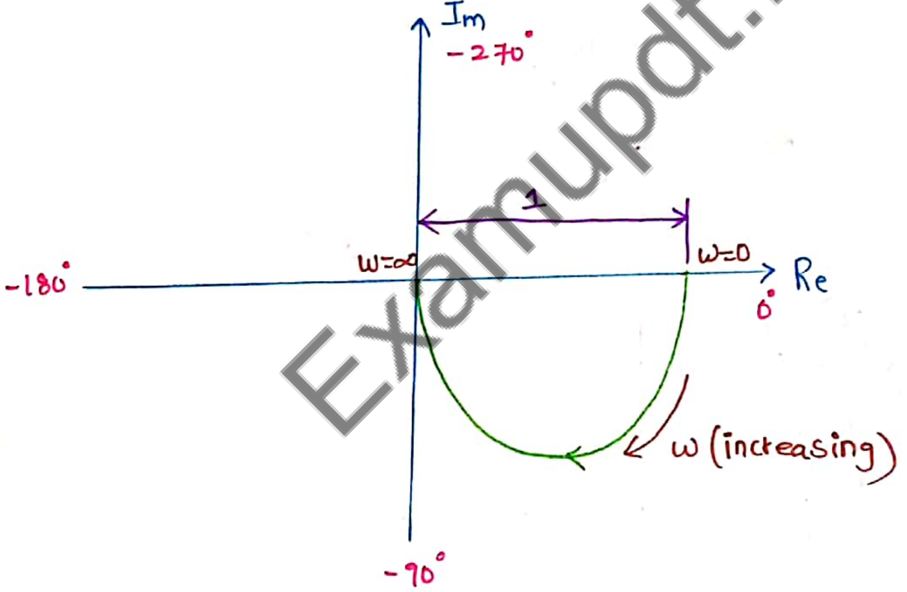
2

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}$$

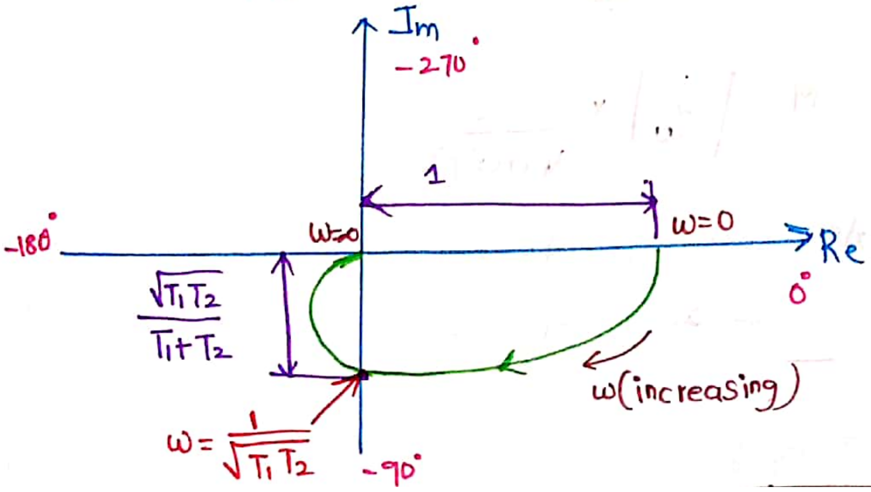
$$G(j\omega) = \frac{1}{1-(0.1\omega^2) + j(0.1\omega)}$$



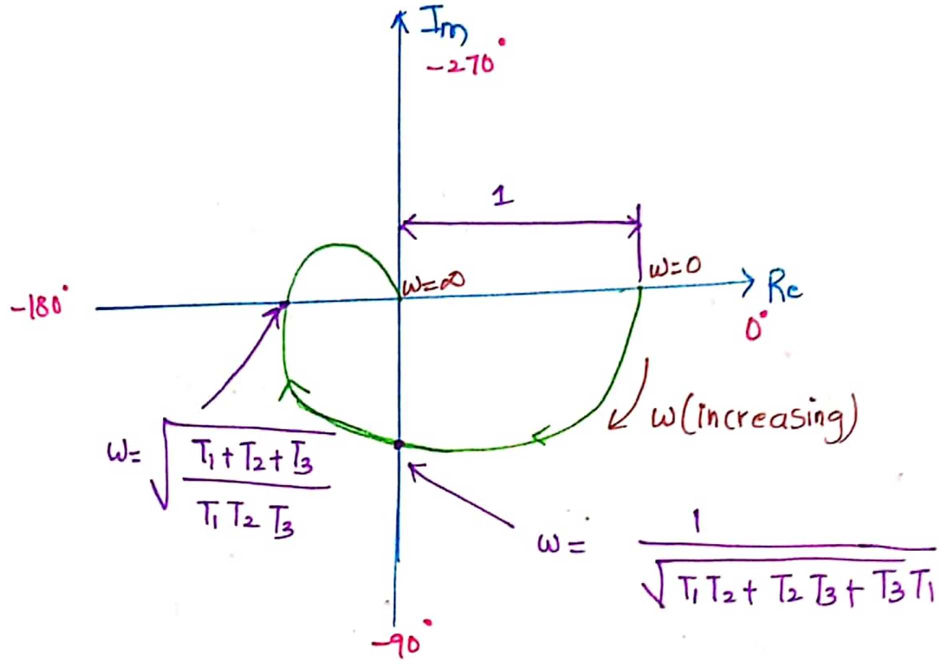
① $G(s) = \frac{1}{1+j\omega T_1}$



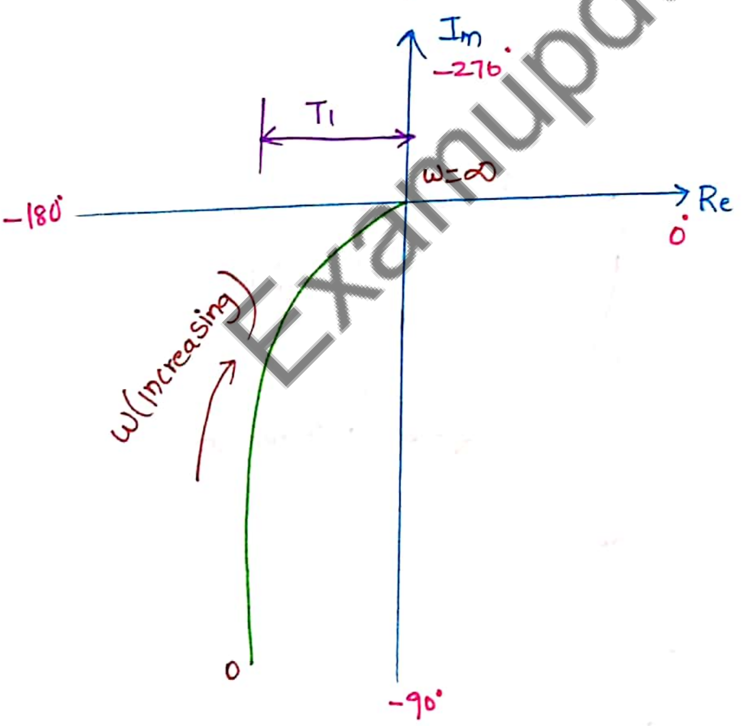
② $G(s) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$



③ $\dfrac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)} = G(s)$



④ $G(s) = \dfrac{1}{j\omega(1+j\omega T_1)}$

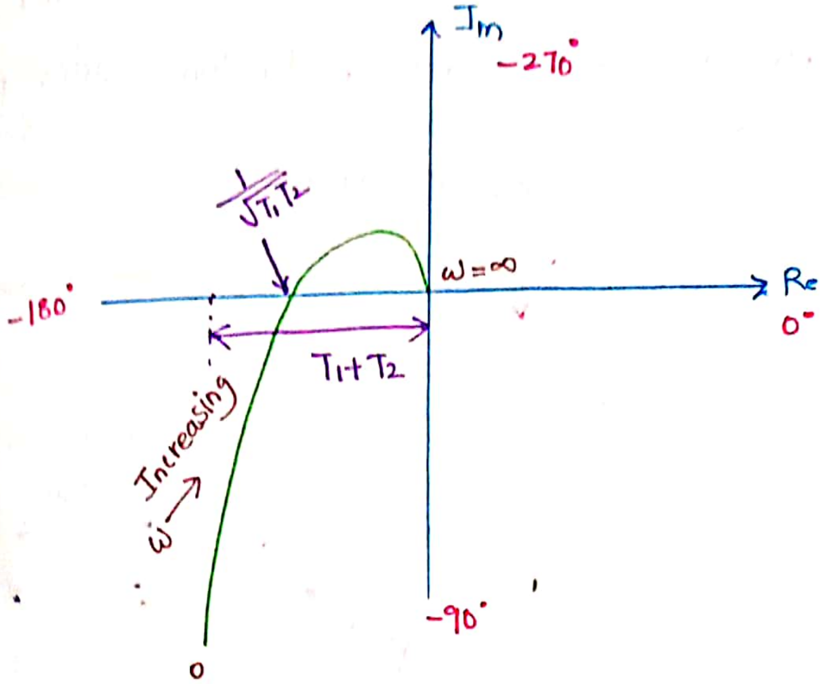


$$M = \left|\frac{1}{\omega}\right| \times \frac{1}{\sqrt{1+\omega^2T^2}}$$

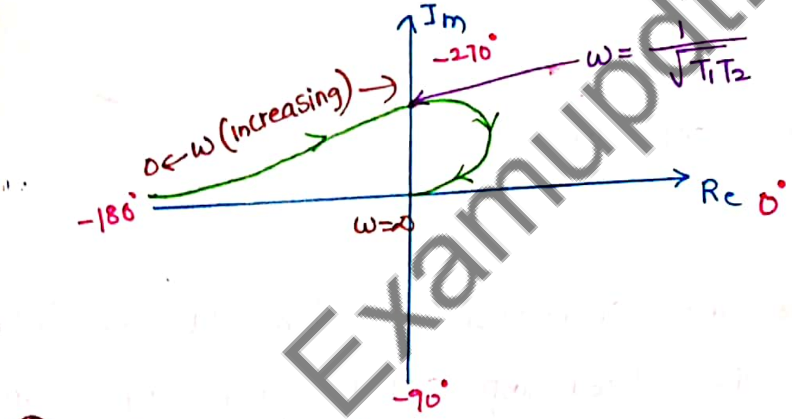
As $\omega = 0$, $M \to \infty$

$\omega = \infty$ $M \to 0$

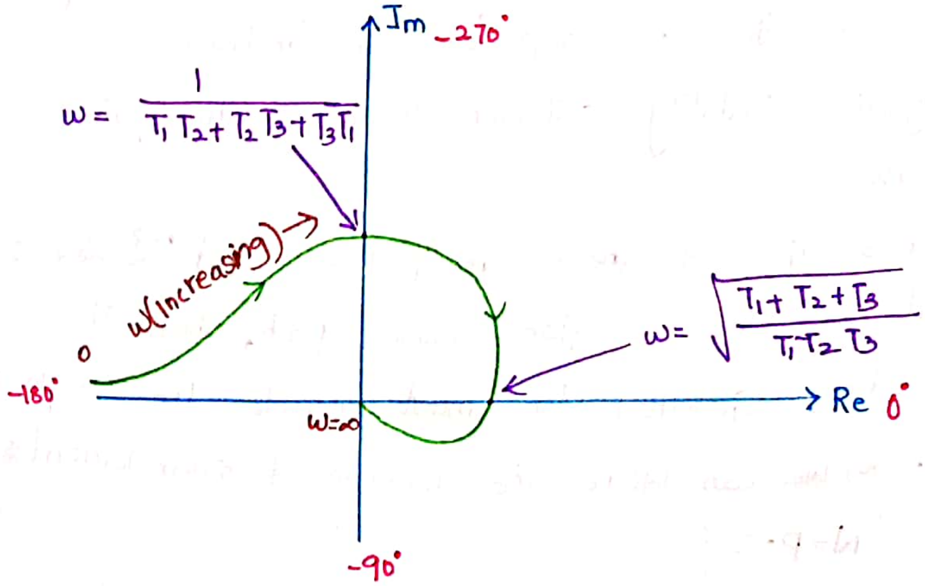
(5) $G(s) = \dfrac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$



(6) $G(s) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$



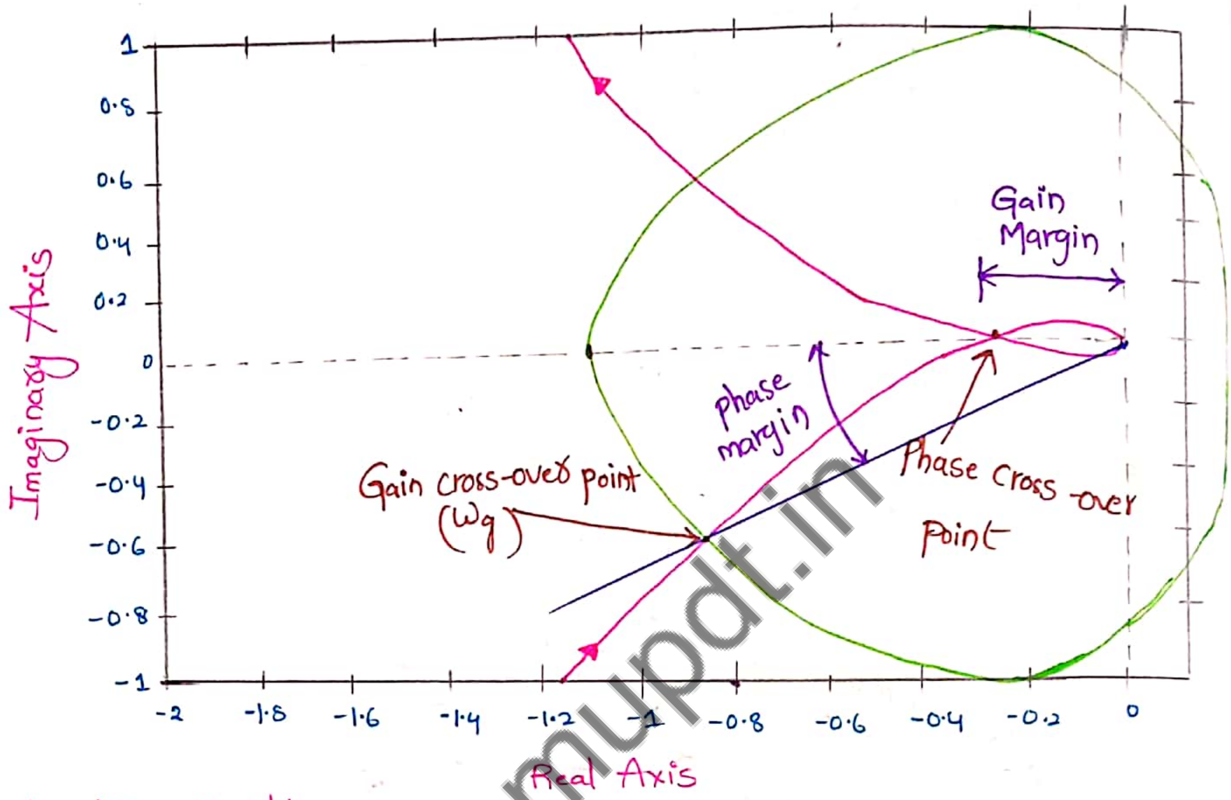
(7) $G(s) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$



1) Addition of pole at origin to Transfer function rotates the polar plot by $-90°$ as $\omega \to \infty$ for $\forall \omega$ and

2) Addition of non-zero pole to Transfer function rotates polar plot as $\omega \to \infty$

Nyquist Diagram



Nyquist stability criterion:

- Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from $-\infty$ to ∞
- Nyquist plots are used to draw the complete frequency response of the open loop transfer function.
- The Nyquist stability criterion works on the principle of argument.
- It states that if there are 'P' poles and 'Z' zeroes are enclosed by the s plane closed path, then the corresponding $G(s)H(s)$ plane must encircle the origin $(P-Z)$ times. So, we can write the number of encirclements N as $\boxed{N = P - Z}$

- If the enclosed 's' plane closed path Contains only poles, then the direction of the encirclement in the G(s) H(s) plane will be opposite to the direction of the enclosed closed path in the s-plane.
- If the enclosed 's' plane closed path contains only zeros, then the direction of the encirclement in the G(s) H(s) plane will be in the Same direction as that of the enclosed closed path in the 's' plane.

Principle of Argument:

- Applying the principle of argument to the entire right half of the 's' plane by selecting it as a closed path. the Nyquist contour.
- Closed loop Control system is stable if all the poles of the closed loop transfer function are in the left half of the 's' plane.
- Poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us concrete these roots of the characteristic equation as follows
  - * The poles of the characteristic equation are same as that of the poles of the open loop transfer function.
  - * The zeros of the characteristic equation are Same as that of the poles of the closed loop transfer function

Nyquist stability criterion:

- We know that the open loop control system is stable if there is no open loop pole in the right half of the 's' plane

i.e., $P=0 \Rightarrow N=-Z$

- We know that the closed loop control system is stable if there is no closed loop pole in the right half of the 's' plane

i.e., $Z=0 \Rightarrow N=P$

- Nyquist stability criterion states the number of encirclements about the critical point $-(1+j0)$ must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane.
- The shift in origin to $(1+j0)$ gives the characteristic equation plane.

$$1+G(s)H(s)=0$$

$$\boxed{G(s)H(s)=-1}$$

Rules for drawing the Nyquist plot:

- Locate the poles and zeros of open loop transfer function $G(s)H(s)$ $G(s)H(s)$ in 's'-plane.
- Draw the polar plot by varying ω from zero to infinity.
- If pole (or) zero present at $s=0$, then varying ω from $0+$ to infinity for drawing polar plot.
- Draw the mirror image of above polar plot for values of ω ranging from $-\infty$ to zero (0^- if any pole (or) zero present at $s=0$).
- The number of infinite radius half circles will be equal to the number of poles (or) zeros at origin.

- The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.
- After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point (-1+j0) lies outside the encirclement, then the closed loop control system is absolutely stable.

Stability Analysis using Nyquist plot

- From the Nyquist plot, we can identify whether the control system is stable, marginally stable (or) unstable based on the values of these parameters.
  - * Gain cross-over frequency and phase cross-over frequency
  - * Gain margin and phase margin

Phase Cross Over Frequency:

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180) is known as the phase cross over frequency. It is denoted by ω_{pc}.

Gain cross over Frequency:

The frequency at which the Nyquist plot is having the magnitude of one is known as the gain cross over frequency. It is denoted by ω_{gc}.

- The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

- If the phase cross over frequency (ω_{pc}) is greater than the gain cross over frequency (ω_{gc}), then the control system is stable.
- If the phase cross over frequency (ω_{pc}) is equal to the gain cross over frequency (ω_{gc}), then the control system is marginally stable.
- If the phase cross over frequency (ω_{pc}) is less than gain cross over frequency (ω_{gc}), then the control system is Unstable

Gain Margin and phase Margin:

Gain Margin:

The Gain Margin (GM) is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$\boxed{\text{Gain margin} = \frac{1}{M_{pc}}}$$

Where M_{pc} is the magnitude at the phase cross-over frequency.

Phase Margin:

The phase Margin (PM) is equal to the sum of 180° and the phase angle ϕ_{gc} at gain cross over frequency

$$\boxed{\text{Phase margin} = 180^\circ + \phi_{gc}}$$

The stability based on the Gain margin and phase margin is described as:

* If the Gain margin is greater than one and the phase margin is positive, the system is stable.
* If the Gain margin is equal to one and the phase margin is zero, the system is marginally stable
* If the Gain margn is less than one and/or phase margin is negative, the system is Unstable

1) Plot the Nyquist plot for the system described by

$$G(s)H(s) = \frac{10}{(s+2)(s+4)}$$

Sol:

$$G(s)H(s) = \frac{10}{(s+2)(s+4)}$$

$$G(j\omega)H(j\omega) = \frac{10}{(j\omega+2)(j\omega+4)}$$

As $\omega=0$,

$$|G(j\omega)H(j\omega)| = \frac{10}{\sqrt{(2)^2}\sqrt{(4)^2}} = \frac{10}{8}$$

$$\therefore \boxed{|G(j\omega)H(j\omega)| = 1.25}$$

$$\angle G(j\omega) = \angle 10 - \angle s+2 - \angle s+4$$

$$= 0 - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

$$= 0 - \tan^{-1}\left(\frac{0}{2}\right) - \tan^{-1}\left(\frac{0}{2}\right)$$

$$\therefore \boxed{\angle G(j\omega) = 0^\circ}$$

As $\omega=\infty$,

$$|G(j\omega)H(j\omega)| = \frac{10}{\infty+\infty}$$

$$\therefore \boxed{|G(j\omega)H(j\omega)| = 0}$$

$$\angle G(j\omega) = \angle 10 - \angle s+2 - \angle s+4$$

$$= 0 - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

$$= 0 - \tan^{-1}\left(\frac{\infty}{2}\right) - \tan^{-1}\left(\frac{\infty}{2}\right)$$

$$= 0 - 90^\circ - 90^\circ$$

$$\therefore \boxed{\angle G(j\omega) = -180^\circ}$$

Angle criterion:

$$0 - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) = -90^\circ$$

$$\tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right) = 90^\circ$$

$$\because \boxed{\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}}$$

$$\frac{\frac{\omega}{2} + \frac{\omega}{4}}{1 - \left(\frac{\omega}{2}\right)\left(\frac{\omega}{4}\right)} = \infty$$

$$1 - \frac{\omega^2}{8} = 0$$

$$\frac{\omega^2}{8} = 1$$

$$\omega^2 = 8$$

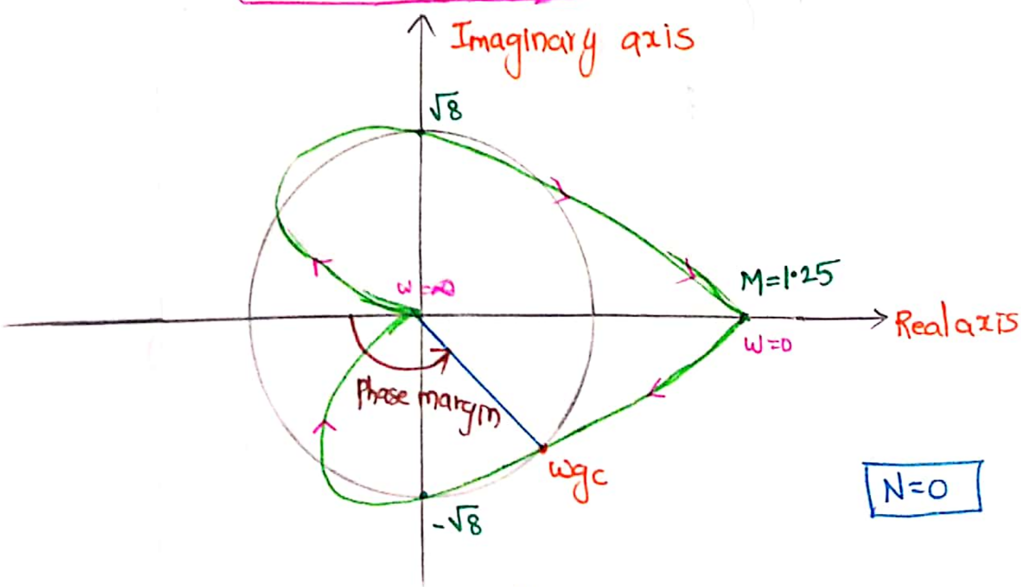
$$\omega = \sqrt{8}$$

$$\omega = 2\sqrt{2}$$

$$\boxed{\omega = 2.828}$$

(Or)

$$\boxed{\omega = \frac{1}{\sqrt{T_1 T_2}} = 2.82}$$



$\boxed{N=0}$

When $\omega = \infty$ then $M = 0$

7) Plot the Nyquist plot for the system described by

$$G(s)H(s) = \frac{10}{s(s+1)(s+4)}$$

Sol:

Given,

$$G(s)H(s) = \frac{10}{s(s+1)(s+4)}$$

In sinusoidal form, $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)(4+j\omega)}$$

$$= \frac{10}{j\omega(1+j\omega)4(1+0.25j\omega)}$$

$$\boxed{G(j\omega)H(j\omega) = \frac{2.5}{j\omega(1+j\omega)(1+j0.25\omega)}}$$

$$\boxed{|G(j\omega)H(j\omega)| = \frac{2.5}{\omega\sqrt{1+\omega^2}\sqrt{1+(0.25\omega)^2}}}$$

$$\angle\phi = \angle 2.5 - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{0.25\omega}{1}\right)$$

$$= 0 - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.25\omega)$$

$$\boxed{\angle\phi = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.25\omega)}$$

ω	0	0.25	0.5	1	2	4	5	10	100	∞
M	∞	9.68	4.43	1.714	0.5	0.10	0.06	0.002	$\simeq 0$	0
ϕ	-90°	-107.6	-123.16	-149.03	-180	-210.96	-220	-242.15	-267.13	-270°

Point of intersection with negative (-ve) real axis (or) $\phi = -180^\circ$

$$\phi = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.25\omega) = -180^\circ$$

$$\tan^{-1}(\omega) + \tan^{-1}(0.25)\omega = 90^\circ$$

Applying tan on both sides

$$\frac{\omega + 0.25}{1 - \omega(0.25\omega)} = \infty$$

Since, $1 - 0.25\omega^2 = 0$

$$0.25\omega^2 = 1$$

$$\omega^2 = \frac{1}{0.25}$$

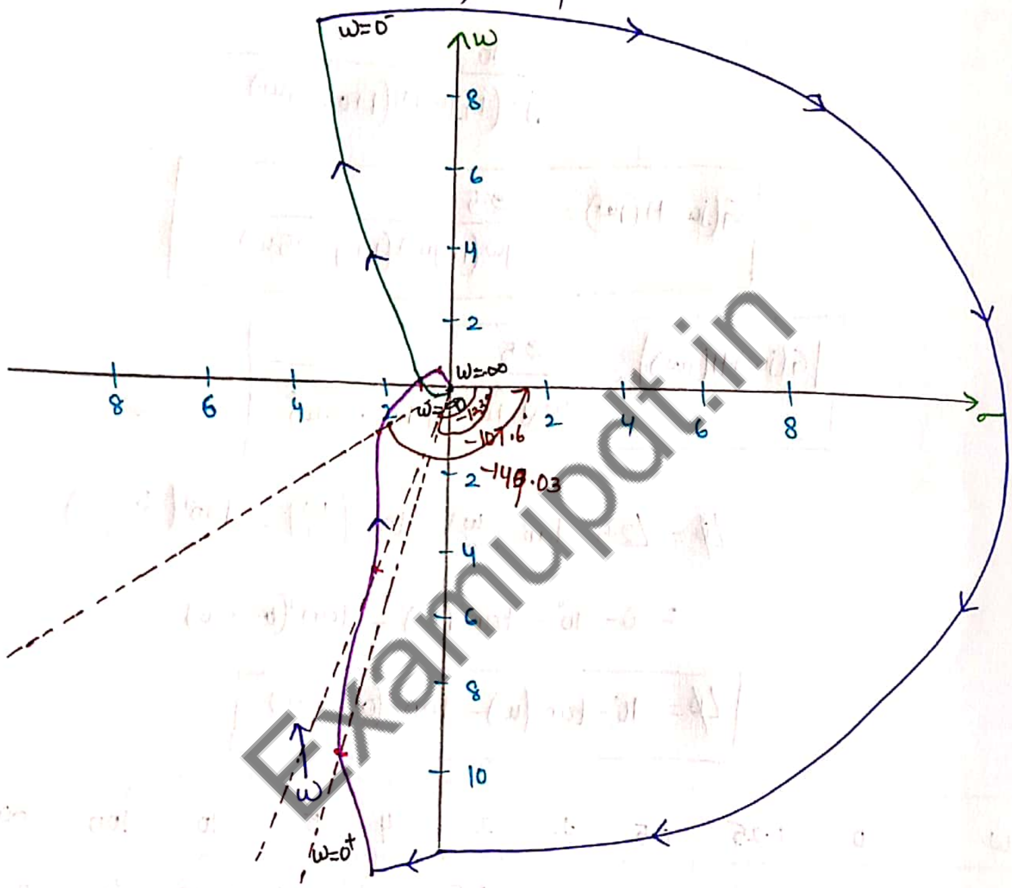
$$\omega^2 = 4$$

$$\boxed{\omega = \pm 2}$$

At $\omega = 2$, $M = 0.42$

$$G(s)H(s) = \frac{10}{s(s+1)(s+4)}$$

$s=0$, $s=-1$, $s=-4$



Here, $N=0$, Since $P=0$, $Z=0$

$$N = P - Z$$

$$0 = 0$$

Therefore, the system is Absolutely state.

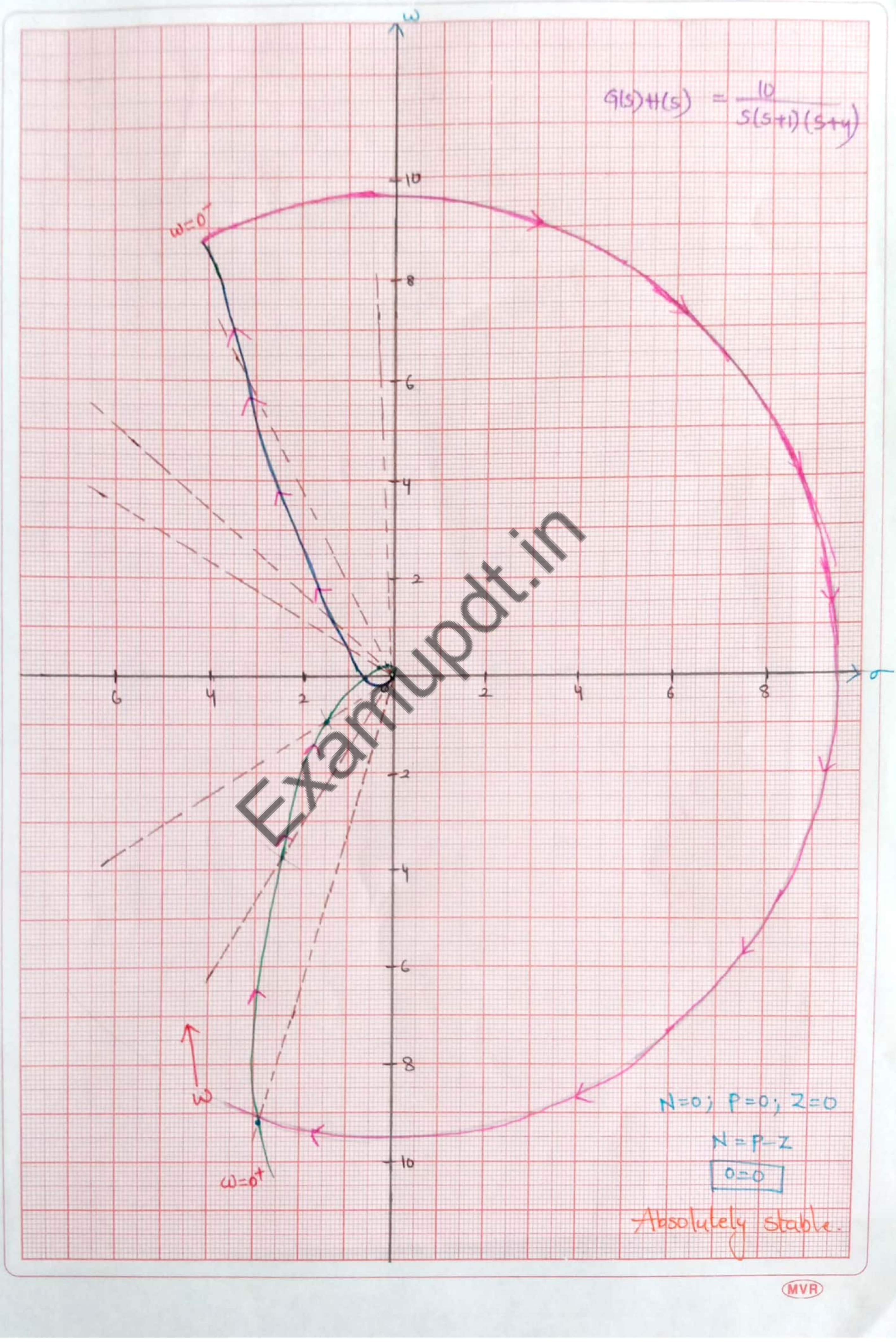
3) Plot the Nyquist plot for the system described by

$$G(s)H(s) = \frac{K}{(s+2)(s-1)}$$

Sol:

Given,

$$G(s)H(s) = \frac{K}{(s+2)(s-1)}$$


G(s)H(s) = 10 / s(s+1)(s+4)
ω
σ
ω=0⁻
ω=0⁺
ω
N=0; P=0; Z=0
N = P−Z
0=0
Absolutely stable.

$S = -2, +1$

$P = 1$; $Z = 0$

$\therefore P - Z = N$

$N = 1 - 0$

$\boxed{N = 1}$

Let $k=1$:

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega+2)(j\omega-1)}$$

$$= \frac{K}{2(1+0.5j\omega)(j\omega-1)}$$

$$G(j\omega)H(j\omega) = \frac{\frac{K}{2}}{(1+0.5j\omega)(j\omega-1)}$$

At $\omega = 0$,

$$M = \left|-\frac{K}{2}\right|$$

$$M = \frac{K}{2}$$

$$\phi = 0 - \tan^{-1}(0.5\omega) - \tan^{-1}(-\omega)$$

$$\phi = 180^\circ$$

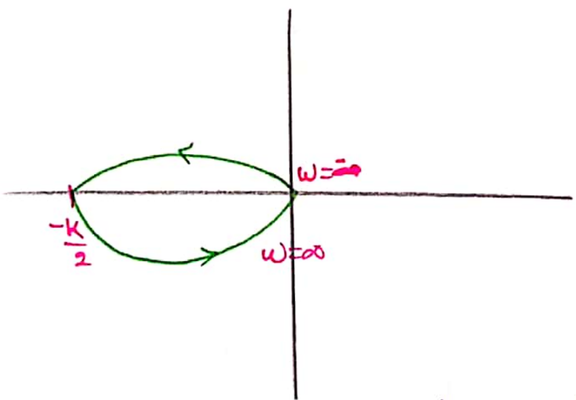
At $\omega = \infty$,

$$M = 0$$

$$\phi = 180^\circ$$

- When point is outside the curve then there is encirclement.
- When point is inside the Cure then there is no encirclement

$\boxed{N=1}$



$-\frac{K}{2} = -1$

$\boxed{K = 2}$

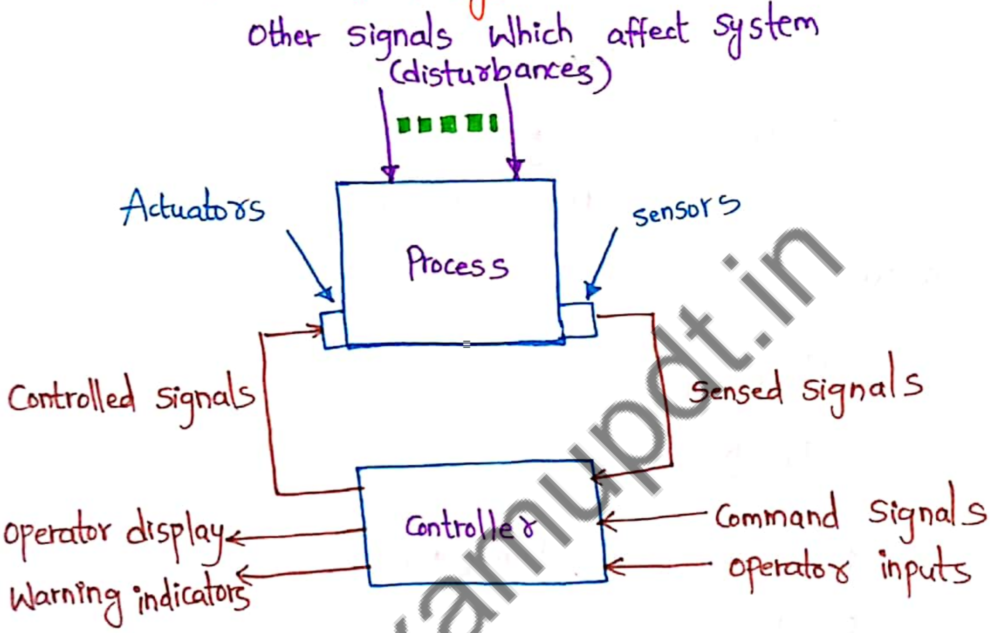
If $K < 2$ then curve encircles and System is stable.

Control Design:

Control Engineering and Control Design:

- The main Concern in Control theory is to study how signals are processed by dynamical systems and how this processing can be influenced to achieve a certain behaviour.
- A control methodology can be defined as the set of techniques and procedures used to construct and/or implement a controller for a dynamical system.

Schematic of a Control system:



Dynamic Systems:

- Found in all major engineering disciplines.
- Merely a mapping from a certain input signal to an output signal.
- Future response of the system is determined by the present state of the system (the initial conditions) and the present input.

Described by three features:

* The first one is the state of a system, which is a representation of all the information about the system at some particular moment of time.
* The second feature is the set of all possible states to which a system can be assigned.
* The third is the state-transition function that is used to update

and change the state from one moment to another.

Control Engineering involves:

- Modelling (or) Identification
- Control configuration
- Control law (or) Controller design
- Control implementation and
- Control system testing and Validation.

Controller Design (or) Control Law:

- The controller (or) Control law can be described as a signal processing to generate appropriate actuator signals based on sensor and other incoming signals.
- To agree with given Specification (required characteristics), the designer must decide exactly how the actuators are to be driven by processing the incoming Sensor signals.
- The important characteristics of control system are stability, Sensitivity, disturbance rejection, steady state accuracy and transient response of a system.

Conventional Controller Design:

- It is necessary to redesign (by modifying the structure or by incorporating additional components) to alter the overall behavior so that the system will behave as desired, and this additional device is called Controller (or) Compensator
- In order to implement conventional controller design, there is a need for the system to be controlled as described with mathematical modelling.

Controller Design Techniques:

- The Nyquist stability criterion is a very valuable tool that determines the degree of stability (or) instability of a feedback control system.
- The Bode-diagram approach is one of the most commonly used methods for the analysis and synthesis of linear feedback

Control systems.

- The Root-locus is a technique, which investigates the movement of the characteristics roots on the s-plane as the system parameters are varied. The knowledge of the location of the closed-loop roots permits the accurate determination of a control system's relative stability and transient performance.

Types of controllers:

Types of compensators:

- Phase lead
- phase lag
- Lag - lead
- P, PI, PD, PID

P - Proportional ; PI - proportional integral ; PD - proportional derivative
PID - proportional integral derivate

Classification of Compensation:

- Series compensation (or) cascade compensation.
- Feedback compensation (or) parallel compensation.
- State feedback compensation
- Feedforward compensation

Performance Specification:

First order system : $G_1(s) = \frac{1}{Ts+1}$

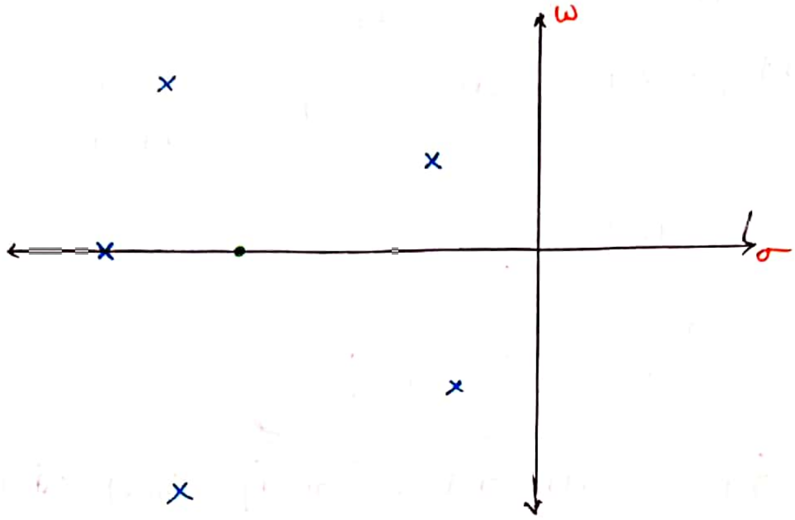
Second order system : $G_2(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

Rise time : $t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \cos^{-1}\xi}{\omega_n\sqrt{1-\xi^2}}$

Settling time : $t_s = \frac{4}{\xi\omega_n}$

Peak Overshoot : $M_p = 100\, e^{\frac{\pi\xi}{\sqrt{1-\xi^2}}}\ \%$

Dominant poles of the system.



- We typically specify the performance of a Control System in terms of these metrics
- Suppose that the plant has order that is relatively high and the closed-loop system has the following pole zero distribution in the s-plane.

What is the Contribution of these poles and zeros to the system?

- The poles closest to the Imaginary axis have the most dominant Contribution and are called the dominant poles.
- Consider the following case,

  A system with an extra pole and zero other than the dominant poles.

  Consider a typical Second order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Effect of addition of poles and zeros:

Consider a typical Second order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For Simplicity,

Let $\xi = 0.5$ and $\omega_n = 1$

$$\boxed{G(s) = \frac{1}{s^2 + s + 1}}$$

Let us start by adding a zero at $s=-a$

To maintain the DC gain at 1,

We add $\left(\frac{s}{a}+1\right)$ to obtain $G_1(s) = \frac{\frac{s}{a}+1}{s^2+s+1}$

$$G_1(s) = \frac{\frac{s}{a}+1}{s^2+s+1}$$

$$\boxed{G(s) = \frac{1}{s^2+s+1} + \frac{1}{a}\cdot\frac{s}{s^2+s+1}}$$

Let $L^{-1}\{G(s)\} = g(t)$ and $L^{-1}\{G_1(s)\} = g_1(t)$ then

$$g_1(t) = g(t) + \frac{1}{a}g(t)$$

The step response,

$$G_1(s) = \frac{\frac{s}{a}+1}{s^2+s+1} \qquad \left[\because \frac{C(s)}{R(s)} = G_1(s)\right]$$

$$C(s) = G_1(s) \times R(s)$$

$$C(s) = G_1(s) \times \frac{1}{s}$$

$$c(s) = \left[G(s) + \frac{1}{a} \times s \times G(s)\right]\frac{1}{s}$$

$$\boxed{c(s) = \frac{G(s)}{s} + \frac{1}{a}G(s)}$$

Let $c_1(s) = \frac{G(s)}{s}$

$$c(s) = c_1(s) + \frac{s}{a}c_1(s)$$

Let $L^{-1}\{c(s)\} = c(t)$

$$L^{-1}\{c_1(s)\} = c_1(t)$$

$$\boxed{c(t) = c_1(t) + \frac{1}{a}c_1(t)}$$

Let $c_2(t) = c_1(t)$

$$\boxed{c(t) = c_1(t) + \frac{1}{a}c_2(t)}$$

Where $c(t)$ is compensated system.

As $c(t) \longleftrightarrow C(s)$

$$C(s) = G_1(s) \times \frac{1}{s}$$

$$G(s) = G(s) \times \frac{1}{s}$$

$c_1(t)$ is the step response of $g(t)$

$c_1(t)$ is orginal system

- The step response of the system is

$$C(s) = G_1(s)\left(\frac{1}{s}\right)$$

$$= \frac{G(s)}{s} + \frac{s}{a}\,\frac{G(s)}{s}$$

$$C(s) = C_1(s) + \frac{s}{a}\,C_1(s)$$

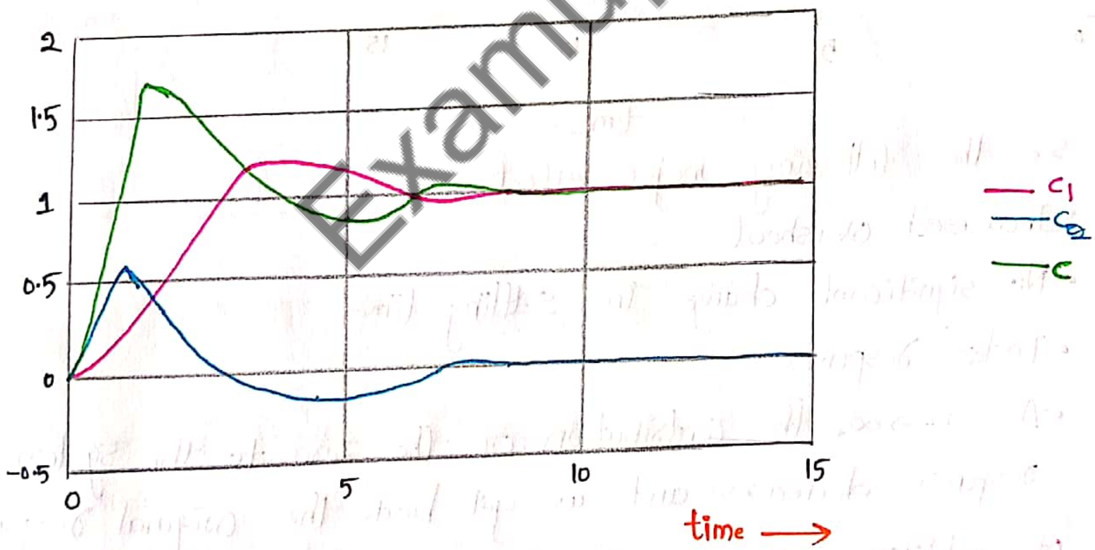
$$\boxed{c(t) = c_1(t) + \frac{1}{a}\,c_2(t)}$$

Where $G(t)$ is the response of $G(s)$

$C(s)$ is the response of $G_1(s)$.

Effect of addition of zero in left half of s-plane

zero at $s = -0.5$

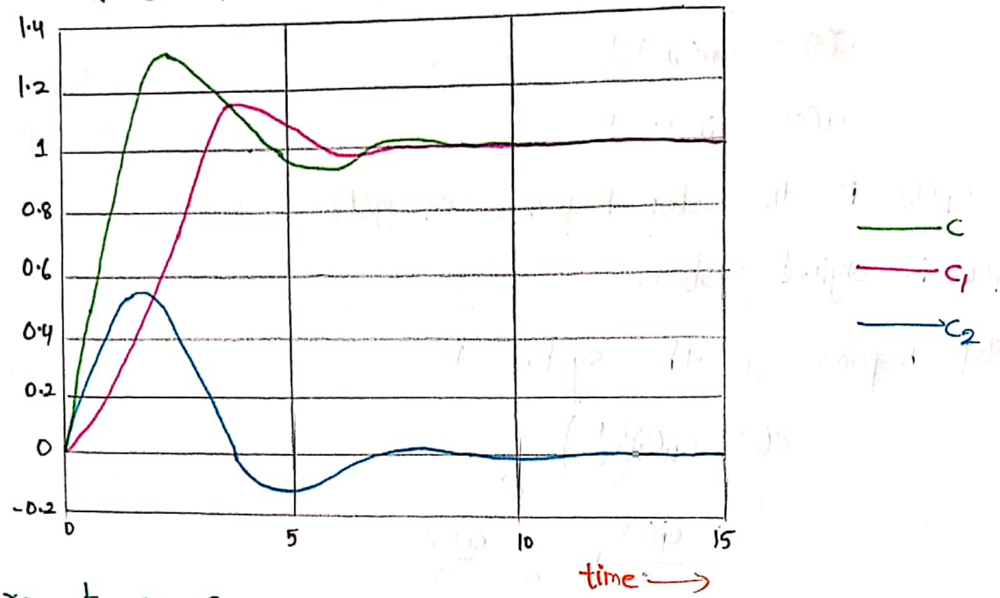


— c

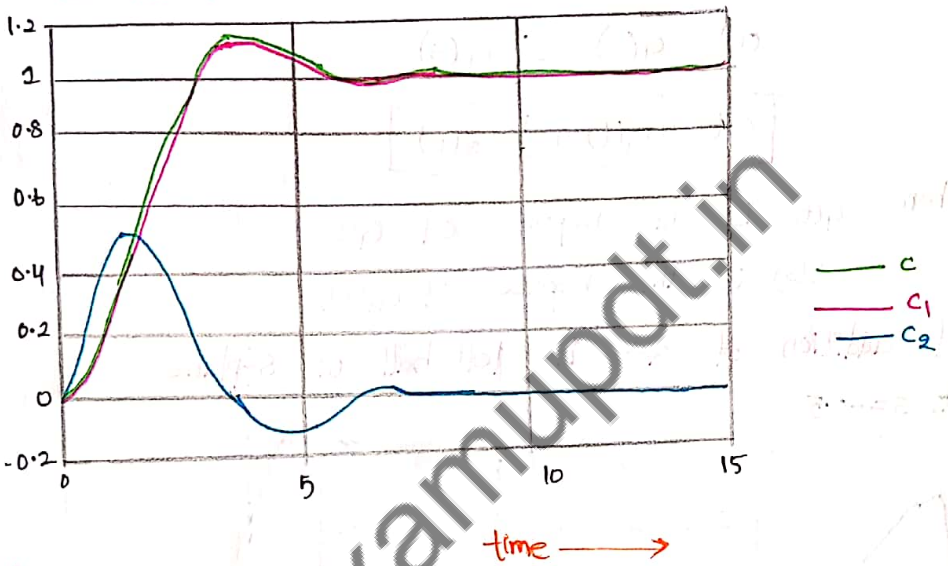
— c_1 (overall response curve)

— c_2

Zero at $s = -1$



zero at $s = -8$



We see the following major effects:

- Increased overshoot
- No significant change in settling time
- Faster response
- As $a \to \infty$, the contribution of the zero to the system response decreases and we get back the original response.

Effect of addition of zero in the right half of s-plane

Let us start by adding a zero at $s = a$

To maintain the DC gain at 1,

We add $\left(1 - \frac{s}{a}\right)$ to obtain $G_1(s) = \dfrac{\left(1 - \frac{s}{a}\right)}{s^2 + s + 1}$

$$G_1(s) = \frac{\left(1 - \frac{s}{a}\right)}{s^2 + s + 1}$$

$$\boxed{G_1(s) = \frac{1}{s^2 + s + 1} - \frac{1}{a}\left(\frac{s}{s^2 + s + 1}\right)}$$

Let $L^{-1}\{G(s)\} = g(t)$ and $L^{-1}\{G_1(s)\} = g_1(t)$

then $g_1(t) = g(t) - \frac{1}{a}\dot{g}(t)$

The step response of the system is

$$C(s) = G_1(s)\left(\frac{1}{s}\right)$$

$$= \frac{G(s)}{s} - \frac{s}{a}\frac{G(s)}{s}$$

$$\boxed{C(s) = C_1(s) - \frac{s}{a}C_1(s)}$$

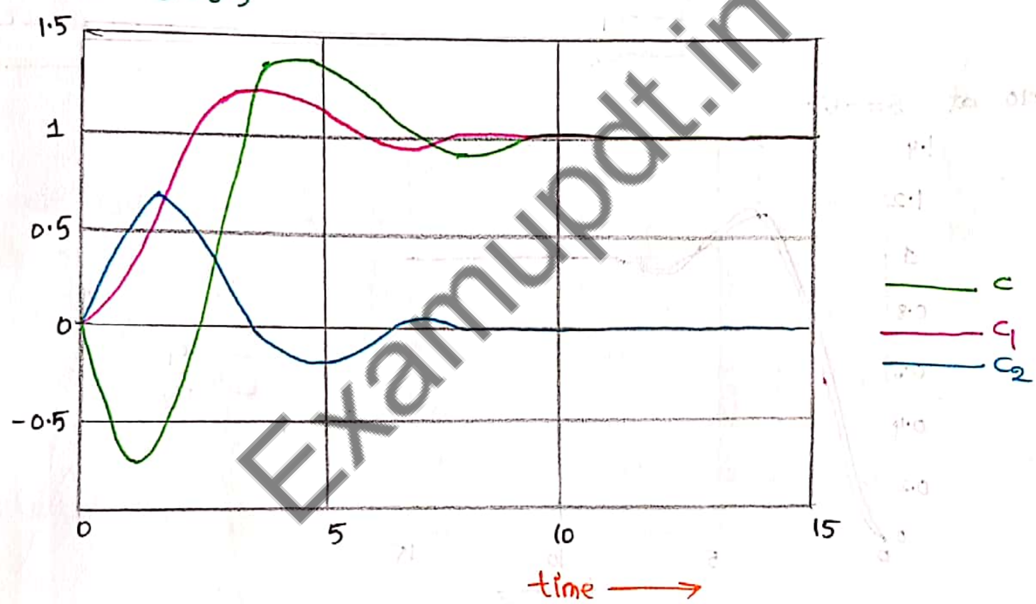
Let $L^{-1}\{C(s)\} = c(t)$ and $L^{-1}\{C_1(s)\} = c_1(t)$

then $c(t) = c_1(t) - \frac{1}{a}\dot{c}_1(t)$

Let $c_2(t) = \dot{c}_1(t)$ then

$$\boxed{c(t) = c_1(t) - \frac{1}{a}c_2(t)}$$

Zero at $s = 0.5$



- The interesting feature is the undershoot in the transient response for small a.
- The undershoot leads to sluggish response with the increase in rise time and delay time.
- Poles and zeros in the RHP are called non-minimum phase poles and zeros
- The term 'non-minimum phase' will be dealt with in more detail when we take up design in the frequency domain.

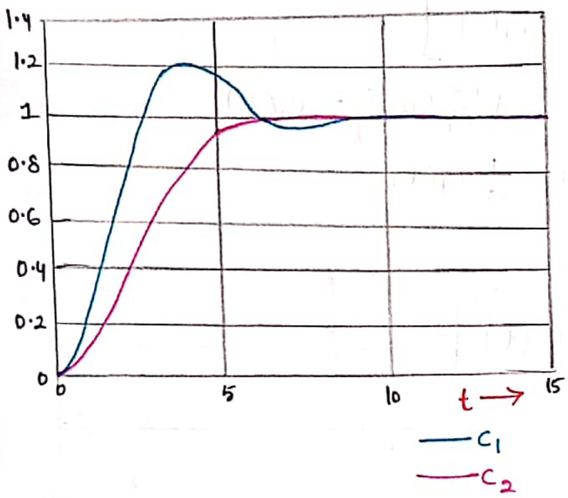
Effect of adding a pole in the left half of s-plane :

- To the second order system considered so far, let us add a pole at $s = -a$

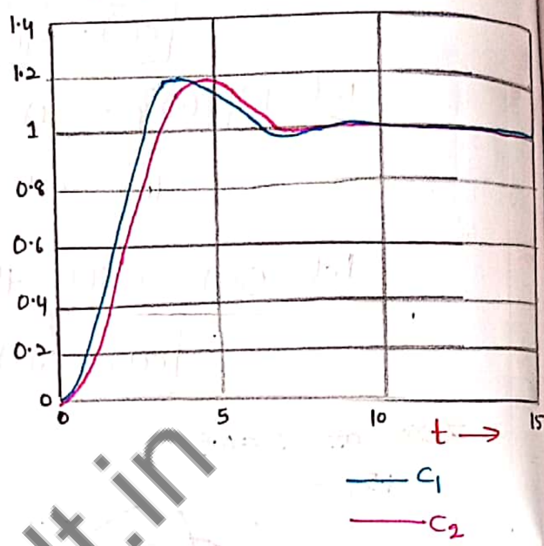
$$\boxed{G_1(s) = \frac{1}{\left(\frac{s}{a}+1\right)} \frac{1}{s^2+s+1}}$$

- step responses of the system for different a.

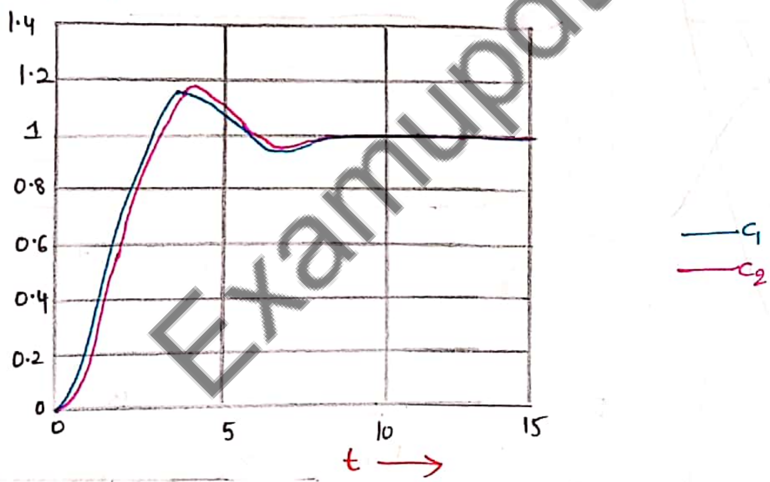
zero at $s = -0.5$



zero at $s = -2.5$



zero at $s = -5$.



- As $a \to \infty$, the contribution of the pole to the system response decreases and we get back the original response.
- In the design process, as a thumb rule, we roughly take $a >> 5$* the real part of the dominant poles.

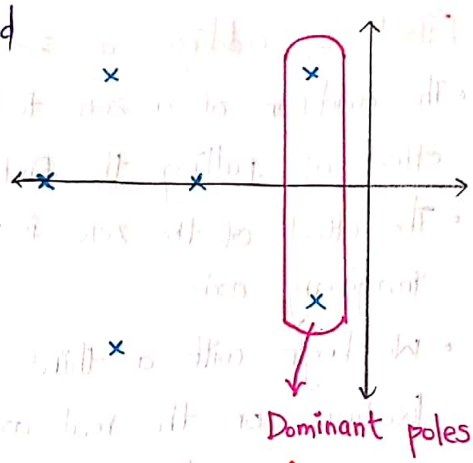
Dominant poles of the system

Dominant poles :

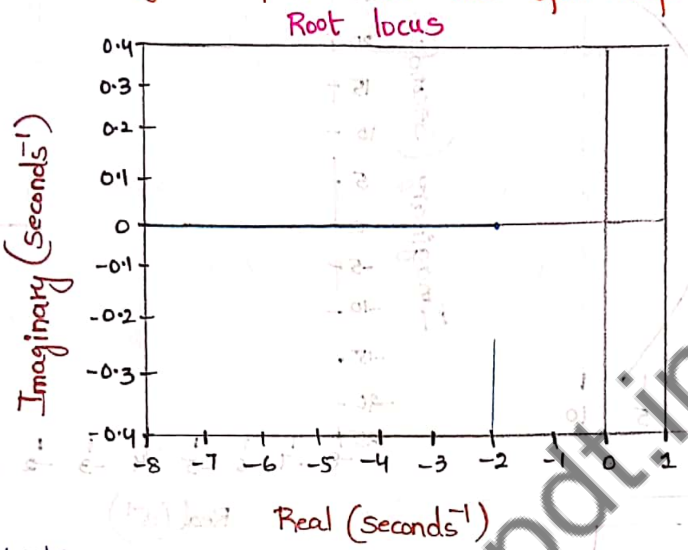
The closed-loop poles that have dominant contribution on the transient response of the system.

- It is often the case that these dominant poles occur in the form complex conjugate poles

- Higher order systems are generally adjusted such that there exists a pair of dominant complex conjugate poles
- It is desirable to have the real parts of the other poles and zeros at least five times farther away than the real parts of the dominant poles

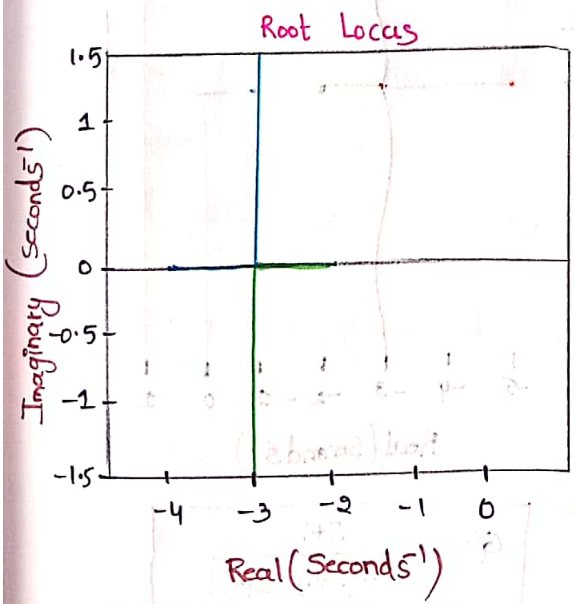


Effect of adding a pole to the open loop transfer function

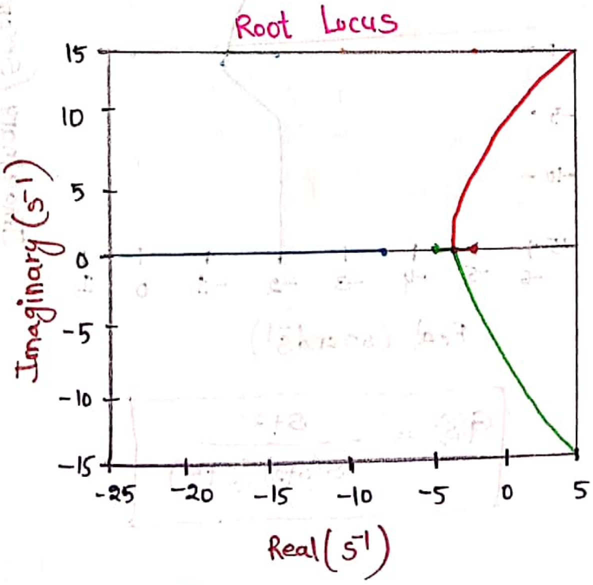


$$G(s) = \frac{1}{s+2}$$

- The addition of a pole to an open-loop transfer function has the effect of pulling the Root Locus to the right.
- Some of the closed-loop poles are nearer to the imaginary axis, which implies the relative stability is reduced.
- We begin with a single pole system and progressively add poles to illustrate these points



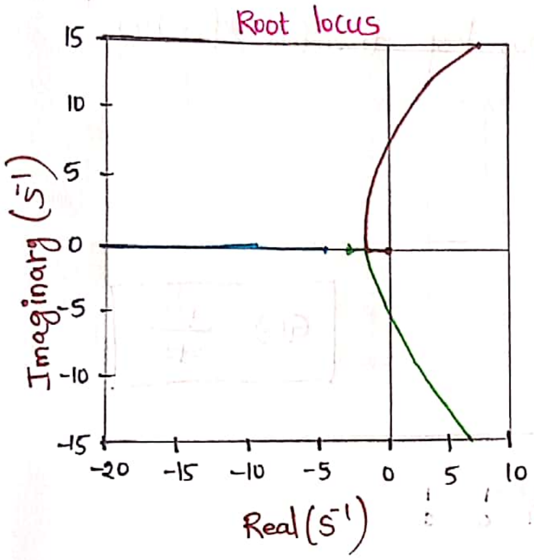
$$G(s) = \frac{1}{(s+2)(s+4)}$$



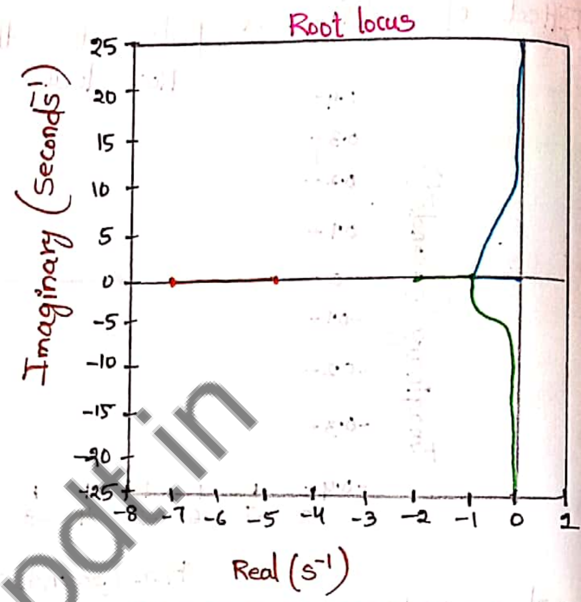
$$G(s) = \frac{1}{(s+2)(s+4)(s+8)}$$

Effect of adding a zero to the open loop transfer function

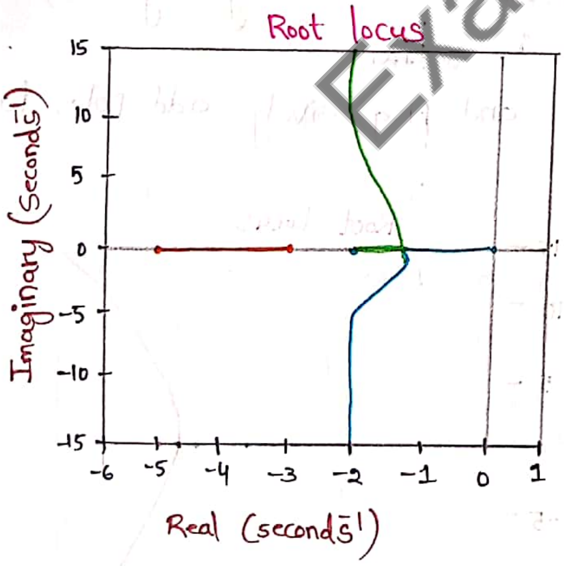
- The addition of a zero to an open-loop transfer function has the effect of pulling the Root Locus to the left.
- The effect of the zero is prominent when it is close to the imaginary axis.
- We begin with a third order system and add a zero at different locations on the real axis to illustrate these points.



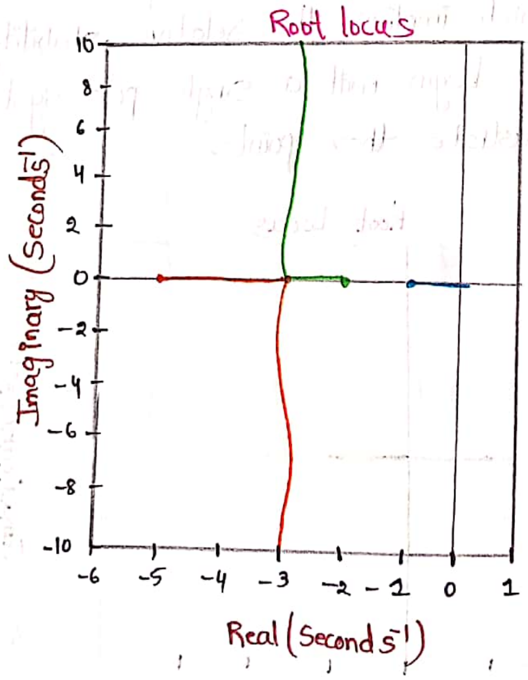
$$G(s) = \frac{1}{s(s+2)(s+5)}$$



$$G(s) = \frac{s+7}{s(s+2)(s+5)}$$



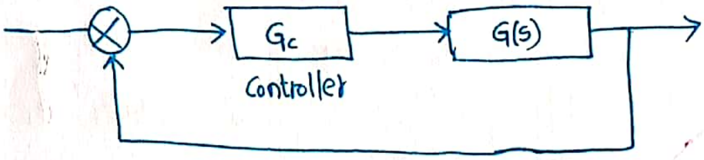
$$G(s) = \frac{s+3}{s(s+2)(s+5)}$$



$$G(s) = \frac{s+1}{s(s+2)(s+5)}$$

Controllers :

- Proportional (K)
- Integral $\left(\frac{K}{S}\right)$
- Derivative (KS)

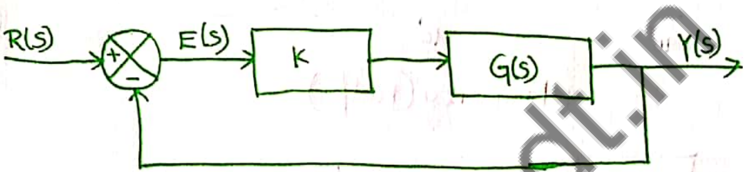


$$\boxed{\frac{C}{R} = \frac{G}{1+G}}$$

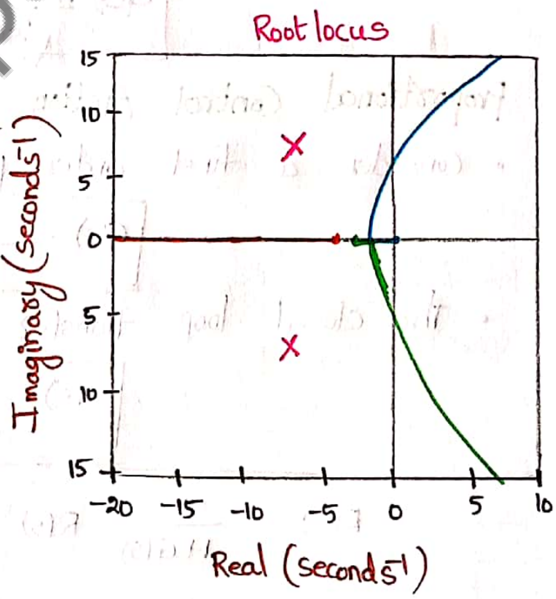
$$\boxed{\frac{C'}{R} = \frac{G_c G}{1+G_c G}}$$

Proportional Control:

The Controller acts on the error between the reference input and the measured output

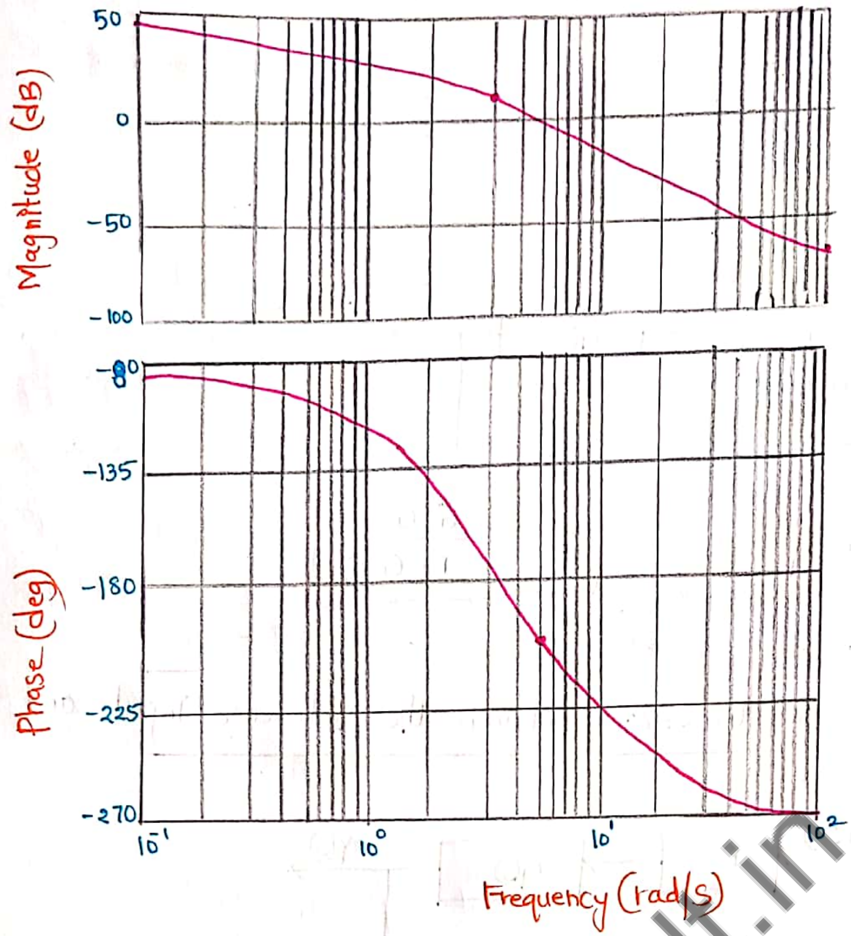


- We have already performed significant analysis with this controller through the Root Locus, Nyquist and Bode plots.
- With the freedom to vary only the gain K we are restricted to move on the Root Locus.
- Works when the poles on the Root Locus meet the specifications.



$$\boxed{G(s) = \frac{1}{s(s+2)(s+5)}}$$

- In the Bode plots, the gain K provided the freedom to shift the magnitude plote up and down.
- In general cases, this may not be sufficient
- If not, more sophisticated controllers are necessary



$$G(s) = \frac{200}{s(s+2)(s+5)}$$

Proportional Control Action

- Consider a first order plant

$$G(s) = \frac{K}{Ts+1}$$

- The closed loop transfer function is

$$C(s) = \frac{K}{Ts+1+K}$$

$$E(s) = \frac{1}{1+G(s)} R(s)$$

$$= \frac{1}{1+\frac{K}{Ts+1}} \frac{1}{s}$$

$$E(s) = \frac{Ts+1}{Ts+1+K} \frac{1}{s}$$

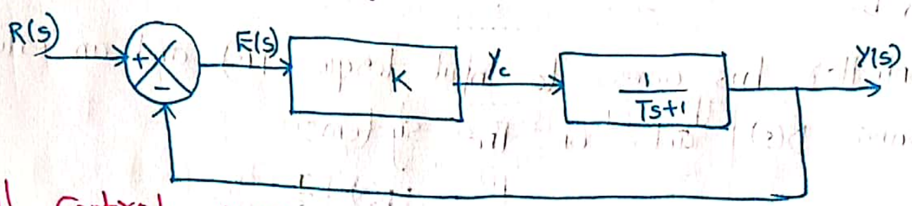
$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{Ts+1}{Ts+1+K}$$

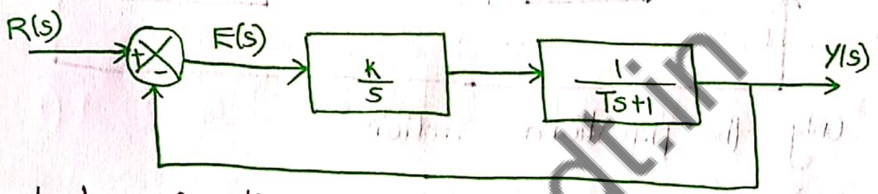
$$\boxed{e_{ss} = \frac{1}{1+K}}$$

- Proportional control improves the time constant from T to $\frac{T}{1+K}$. However, there is steady state error.
- Steady state error can be reduced by choosing K. But high value of K can destabilize the system.



Integral Control Action

Choose a controller that acts on the accumulated error.



The output of the controller is,

$$Y_c(s) = \frac{K}{s} E(s)$$

$$\boxed{y(t) = K \int_{-\infty}^{t} e(t)\, dt}$$

$$G(s) = \frac{K}{s(Ts+1)}$$

$$E(s) = \frac{1}{1+G(s)} R(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{K}{s(Ts+1)}}$$

$$\frac{E(s)}{R(s)} = \frac{s(Ts+1)}{s(Ts+1)+K}$$

$$\boxed{E(s) = \frac{s(Ts+1)}{s(Ts+1)+K} \cdot \frac{1}{s}}$$

$$\boxed{E(s) = \frac{s(Ts+1)}{s(Ts+1)+K} \cdot \frac{1}{s}}$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{s^2(Ts+1)}{s(Ts+1)+K} \cdot \frac{1}{s}$$

$$\boxed{e_{ss} = 0}$$

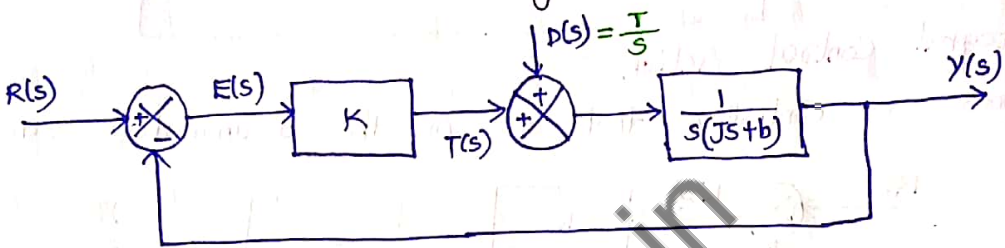
With an integrator, the steady state error is zero.

- Order of the system has increased from 1 to 2.

- For higher order systems, addition of an extra pole may lead to instability
- Integral controller cannot be used alone

Example:

- Consider the second order system $G(s) = \frac{1}{s(Js+b)}$ subjected to external disturbances.
- Models a rotational element with moment of inertia J and viscous friction b.
- The controller has access to input torque $T(s)$ and an external disturbance $D(s)$ acts on the systems



- Consider only the proportional action
- Setting $R(s) = 0$,

  Let us obtain the transfer function from $D(s)$ to $Y(s)$

$$\frac{Y(s)}{D(s)} = \frac{1}{Js^2+bs+K}$$

$$\frac{E(s)}{D(s)} = \frac{-Y(s)}{D(s)} = -\frac{1}{Js^2+bs+K}$$

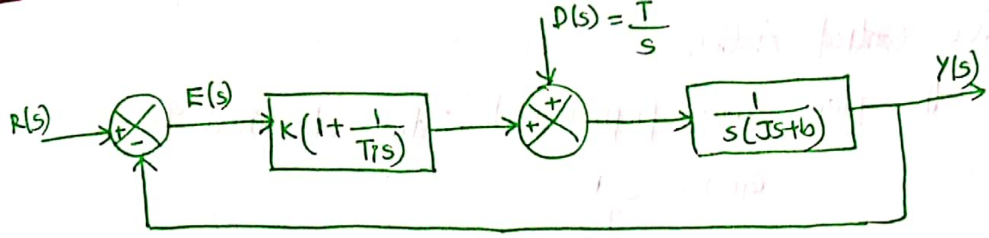
The steady state error is $e_{ss} = \lim_{t\to\infty} e(t)$

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$e_{ss} = \lim_{s\to 0} -\frac{s}{Js^2+bs+K}\,\frac{T}{s}$$

$$\boxed{e_{ss} = -\frac{T}{K}}$$

- Steady state error is finite and can be reduced by increasing K.
- However, for higher order systems, increasing K could lead to instability



- Considering a proportional + Integral Control action
- The transfer function for the disturbance signal is

$$\frac{Y(s)}{D(s)} = \frac{\frac{1}{s(Js+b)}}{1+\frac{1}{s(Js+b)} \times K\left(1+\frac{1}{T_i s}\right)}$$

$$= \frac{T_i s}{s(Js+b)+K(T_i s+1)}$$

$$\boxed{\frac{Y(s)}{D(s)} = \frac{s}{Js^3+bs^2+Ks+\frac{K}{T_i}}}$$

$$\frac{E(s)}{D(s)} = \frac{-Y(s)}{D(s)}$$

$$\boxed{\frac{E(s)}{D(s)} = \frac{-s}{Js^3+bs^2+Ks+\frac{K}{T_i}}}$$

- Before we apply final value theorem, we must ensure that the system is stable. The constant K and T_i must be chosen such that the roots have negative real parts.

The steady state error is $e_{ss} = \lim_{t\to\infty} e(t)$

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$e_{ss} = \lim_{s\to 0} \frac{-s}{Js^3+bs^2+Ks+\frac{K}{T_i}} \cdot \frac{T}{s}$$

$$\boxed{e_{ss} = 0}$$

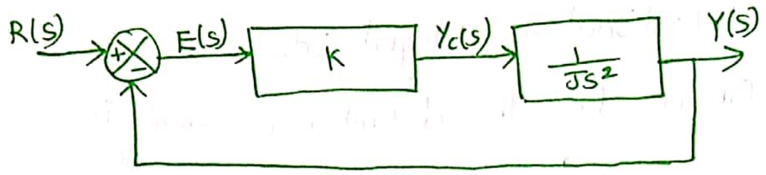
- The proportional + integral controller eliminates the steady state error.
- The proportional controller ensures the stability while the integral term eliminates the error.

$K \to$ Stability

Derivative Control Action

Consider the plant with proportional Control described by

$$G(s) = \frac{1}{Js^2}$$



- The closed loop transfer function is

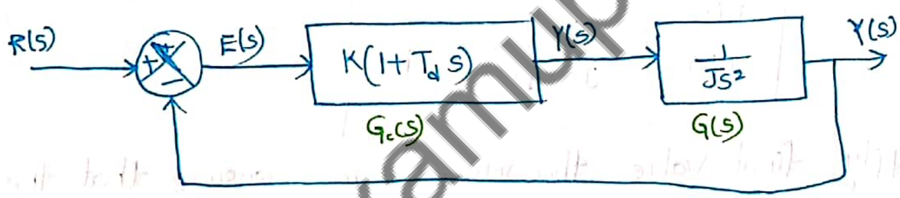
$$C(s) = \frac{K}{Js^2+K}$$

- The poles of the characteristics equation are on the imaginary axis and the system response is purely oscillatory
- The oscillations can be damped using a derivative term in Controller.

Proportional + Derivative Control action

- The Controller has the following structure $G_c(s) = K(1+T_d s)$

The output of the Controller is,

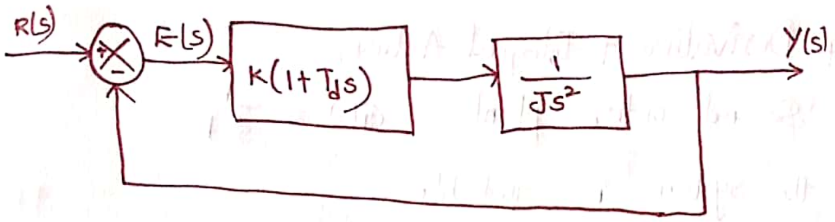


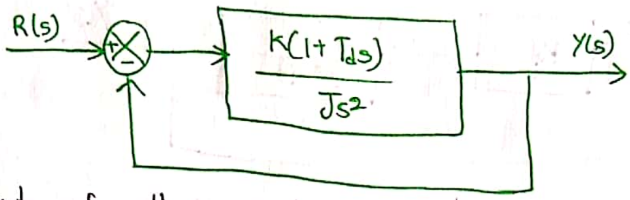
$$\frac{E(s)}{R(s)} \Rightarrow Y_c(s) = G_c(s)\, E(s)$$

$$Y_c(s) = KE(s) + KT_d s\, E(s)$$

$$\boxed{Y_c(t) = Ke(t) + KT_d\, e'(t)}$$

- The derivative Controller acts on the rate of change of error signal. The output is proportional to the rate of change of the error signal.
- The derivative term anticipates the large overshoot, when the rate of change of error is high and takes corrective action.
- The effect can be noticed in the improved damping in the system.





- The output of the system is

$$\frac{Y(s)}{R(s)} = \frac{\frac{K(1+T_d s)}{Js^2}}{1+\frac{K(1+T_d s)}{Js^2}}$$

$$\frac{Y(s)}{R(s)} = \frac{K(1+T_d s)}{Js^2+KT_d s+K}$$

$$\boxed{Y(s) = \frac{K(1+T_d s)}{Js^2+KT_d s+K} R(s)}$$

- For positive values if K, T_d, J the system is always stable.

As $Js^2+KT_d s+K=0$ [characteristic equation]

$$s^2+\frac{KT_d}{J}s+\frac{K}{J}=0$$

The roots

$$s = \frac{-\frac{KT_d}{J} \pm \sqrt{\left(\frac{KT_d}{J}\right)^2 - 4\frac{K}{J}}}{2}$$

$$= \frac{-KT_d}{2J} \pm \sqrt{\frac{K^2T_d^2}{4J^2} - \frac{4K}{4J}}$$

$$\boxed{s = \frac{-KT_d}{2J} \pm \sqrt{\frac{K}{J}\left(\frac{KT_d^2}{4J}-1\right)}}$$

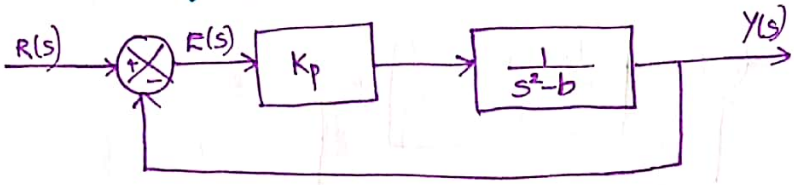
- The damping term in the characteristic equation is due to the derivative control.
- For proportional + derivative control action,

1) Damping is improved
2) System is stable
3) Sustained oscillations becomes damped oscillations.

Proportional + Derivative + Integral Action

Consider a Second order plant $G(s) = \frac{1}{s^2-b}$

with $b>0$, the system is unstable



$$\frac{Y(s)}{R(s)} = \frac{\frac{K_p}{s^2-b}}{1+\frac{K_p}{s^2-b}}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K_p}{s^2-b+K_p}}$$

As $s^2-b+K_p = 0$ [characteristic equation]

$s^2 = b-K_p$

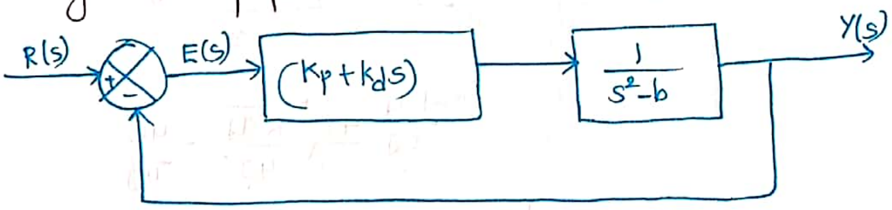
$$\boxed{s = \pm\sqrt{b-K_p}}$$

- With a proportional Controller, the closed loop transfer function is

$$\boxed{Y(s) = \frac{K_p}{s^2-b+K_p}}$$

- System Cannot be stabilized. At best it can be marginally stable.

Considering the proportional + Derivative Controller.



The closed loop transfer function is

$$Y(s) = \frac{K_p+K_d s}{s^2+K_d s+K_p-b}$$

$$Y(s) = \frac{\frac{K_p+K_d s}{s^2-b}}{1+\frac{K_p+K_d s}{s^2-b}}$$

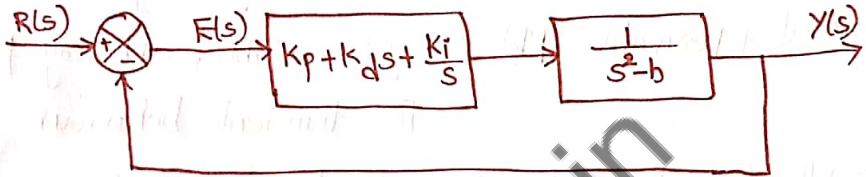
$$\boxed{Y(s) = \frac{K_p+K_d s}{s^2+K_d s+K_p-b}}$$

Characteristic equation $s^2 + k_d s + k_p - b = 0$

The roots $\boxed{s = \dfrac{-k_d \pm \sqrt{k_d^2 - 4(k_p - b)}}{2}}$

- For $k_d > 0$, $k_p > b$, the system is stable and arbitary pole placement is possible
- The closed loop poles can now be placed in the s-plane to meet any given specification.
- The Dc Gain in the system is not equal to 1 and there will be steady state error.

Now Consider adding an Integral term also to the Controller



$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K_p + K_d s + \frac{K_i}{s}}{s^2 - b}}{1 + \dfrac{K_p + K_d s + \frac{K_i}{s}}{s^2 - b}}$$

$$\frac{Y(s)}{R(s)} = \frac{K_p + K_d s + \frac{K_i}{s}}{s^2 + K_d s + \frac{K_i}{s} + K_p - b}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + K_d s^2 + (K_p - b)s + K_i}}$$

Adding an Integral term also, the Dc Gain of the system is

$$e_{ss} = \lim_{t \to \infty} y(t)$$

$$e_{ss} = \lim_{s \to 0} sY(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{s(K_d s^2 + K_p s + K_i)}{(s^3 + K_d s^2 + (K_p - b)s + K_i)} \frac{1}{s}$$

$$\boxed{e_{ss} = 1}$$

- The integral controller provides for perfect tracking, rejects Constant disturbance inputs.

Using PID Control, we can

- Stabilize an unstable Second order System.
- Through the multiple degrees of freedom offered by PID Control, we can place the poles arbitarily
- System has perfect tracking of step signals.
- System rejects Constant disturbance inputs.

Proportional (P) → Adjust DC Gain improves time response and decreases error (M_p)

Proportional + Integral (PI) → error is eliminated then error = 0 i.e $\boxed{e_{ss} = 0}$

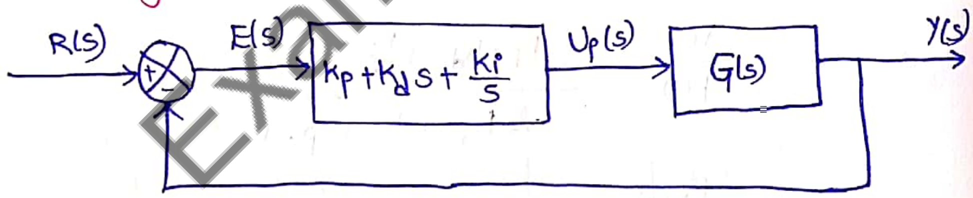
Proportional + Derivative (PD) → System stable, arbitary pole placement for transient behaviour.

PI → When steady state error is to be zero

PD → When transient (time) response is improved

PID → Better time response and good tracking $\boxed{e_{ss}=0}$

Proportional Integral Derivative Actions

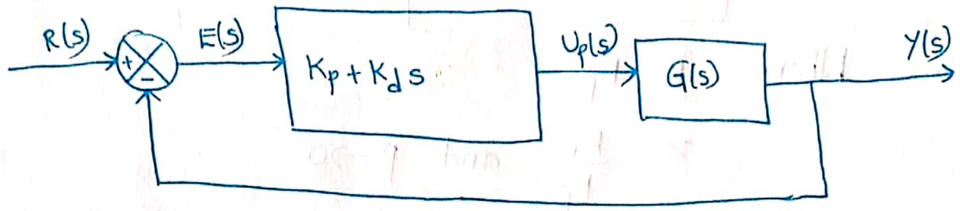


$$U_p(s) = \left(K_p + K_d s + \frac{K_i}{s}\right) E(s)$$

$$\boxed{U_p(t) = K_p e(t) + K_d e'(t) + K_i \int_0^t e(\tau)\, d\tau}$$

P Control	PD Control	PI Control	PID Control
• Simplest to Implement • May not be sufficient to meet the design requirements	• Stability and improved time • provides more control over pole locations	• Perfect tracking of Constant reference inputs • Rejection of Constant disturbance inputs	• stability • Control over pole locations • Perfect tracking of constant reference inputs • Constant disturbance rejection

Derivative and Proportional + Derivative Controller



$$U_p(s) = (K_p + K_d s)\,E(s) \longleftrightarrow u_p(t) = k_p e(t) + k_d \dot{e}(t)$$

- The PD controllers acts on the error and the rate of change of error.
- It improves the stability as we are effectively adding a zero to the open loop transfer function in the left half of the s-plane.
- Derivative term is somewhat anticipatory (or) predictive and its effect was explicitly seen in the improved damping in the system.
- Just the proportional (or) derivative term is not sufficient and both the terms are required in most cases.

Compensators

- Lag Compensator
- Lead Compensator
- Lag-Lead Compensator

Lag Compensation

- PI Controller provides for a perfect tracking of constant reference inputs and helps in rejecting constant external disturbances.
- PI Controller is not a stable system leading to issues such as integrator wind up and also the possibility of destabilizing the closed loop system.
- If small steady state error are acceptable, perfect tracking is not necessary.
- These observations lead us to an approximation in the PI Controller called the Lag Compensator

$$G_c(s) = K_p + \frac{K_i}{s}$$

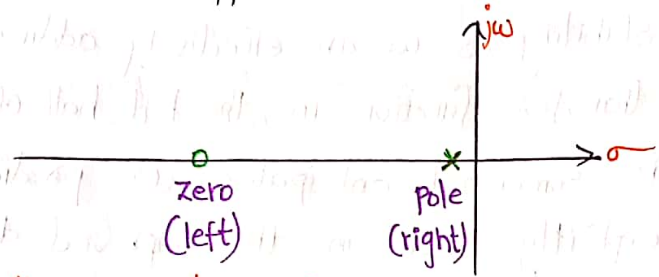
$$G_c(s) = \frac{K_i + K_p s}{s} \approx \frac{K_i + K_p s}{s+p} \cong K\frac{s+z}{s+p}$$

$$\boxed{G_c(s) \approx K\frac{s+z}{s+p}}$$

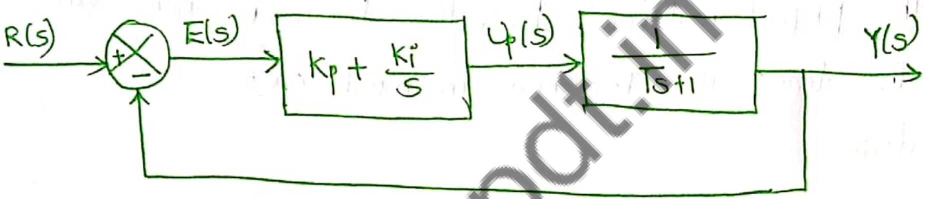
Where $K = K_p$

$z = \frac{K_i}{K_p}$ and $p \to 0$

- Instead of a pole at the origin and a zero in the LHP, we now have a pole and zero in the LHP and the controller is stable. This approximation is called the lag compensator

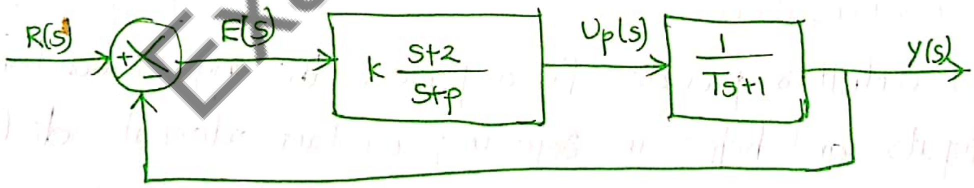


P.I Controller $\approx$ Lag Compensator



$$K_p = \lim_{s \to 0} G(s)\, G_c(s) = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0$$



$$K_p = \lim_{s \to 0} G(s)\; G_c(s) = K\frac{z}{p}$$

$$e_{ss} = \frac{1}{1+K\frac{z}{p}} \cong 0$$

The steady state error in a lag compensator is zero but z and p can be placed to achieve very small error

$$G_c(s) = K\frac{(s+z)}{(s+p)}$$

In sinusoidal $s = j\omega$

$$G_c(j\omega) = K\frac{(j\omega+z)}{(j\omega+p)}$$

$$\phi = \angle K + \angle j\omega + z - \angle j\omega + p$$

$$= 0 + \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right)$$

(small)

$$\boxed{\phi = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right)}$$

Lead Compensation

Derivative Controller with a casual approximation is

$$T_d(s) = \frac{K_d s}{\frac{s}{p} + 1}$$

$$\boxed{T_d(s) = \frac{K_d p s}{s + p}}$$

Similar approximation of the PD Controller can be expressed as

$$G_c(s) = K_p + K_d s \cong \frac{p(K_p + K_d s)}{s + p} = K\frac{s + z}{s + p}$$

Where $K = pK_d$ and

$$z = \frac{K_p}{K_d}$$

- This Controller is stable and less susceptible to high frequency noise.
- It is to be noted that both the pole and zero are in the LHP of s-plane with the pole farther from the zero and the imaginary axis making its effect negligible.
- This approximation is called a Lead Compensator.

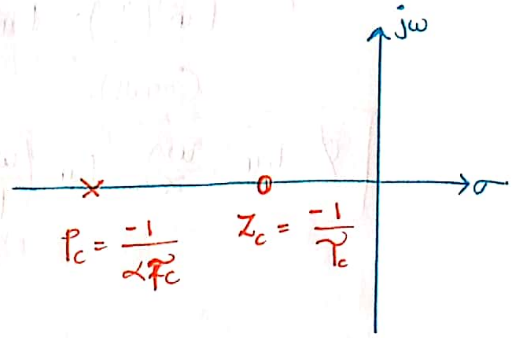
Realization of Compensators:

1) Lead Compensator
2) Lag Compensator
3) Lag-Lead Compensator.

Pole - Zero Configuration :

Lead Network :

$$P_c = \frac{-1}{\alpha T_c} \; ; \; Z_c = \frac{-1}{T_c}$$



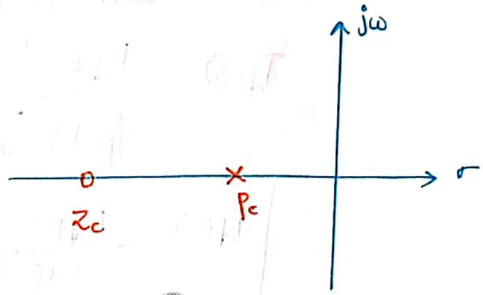
$$\alpha = \frac{Z_c}{P_c} < 1$$

$$\boxed{Z_c > 0}$$

Lag Network :

$$Z_c = \frac{-1}{T_c} \; ; \; P_c = \frac{-1}{\beta T_c}$$

$$\beta = \frac{Z_c}{P_c} > 1$$

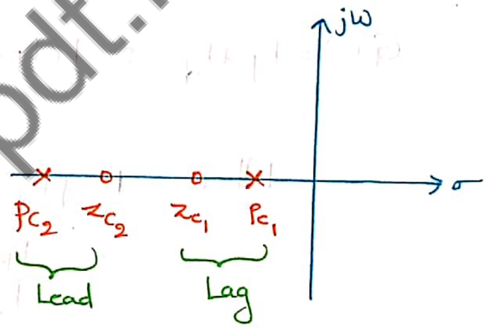


$\therefore$

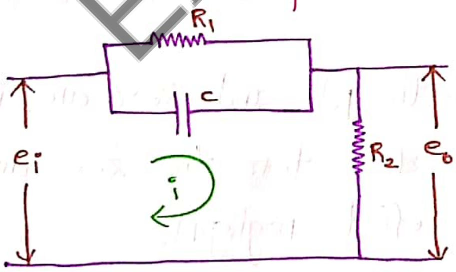
Lag - Lead Network :

Lead - P_{c_2}, Z_{c_2}

Lag - Z_{c_1}, P_{c_1}



Realization of Lead Compensator :



$R_1 \| C$ then $Z = \dfrac{\frac{R_1}{Cs}}{1 + \frac{1}{Cs}}$

$$E_i(s) = \left[\frac{\frac{R_1}{Cs}}{R_1 + \frac{1}{Cs}}\right] I(s) + R_2 I(s)$$

$$E_o(s) = R_2 I(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 I(s)}{\left(R_2 + \frac{\frac{R_1}{Cs}}{R_1 + \frac{1}{Cs}}\right) I(s)}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{\left(R_2 + \frac{R_1}{R_1Cs+1}\right)}$$

$$= \frac{R_2(1+R_1Cs)}{R_2(1+R_1Cs)+R_1}$$

$$= \frac{1+R_1Cs}{1+R_1Cs+\frac{R_1}{R_2}}$$

$$= \frac{1+R_1Cs}{R_1Cs+\left(\frac{R_1+R_2}{R_2}\right)}$$

$$= \frac{R_1C\left(s+\frac{1}{R_1C}\right)}{R_1C\left[s+\left(\frac{R_1+R_2}{R_1R_2C}\right)\right]}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C\left(\frac{R_2}{R_1+R_2}\right)}}}$$

Time constant, $T_c = R_1C$

$$\frac{E_o(s)}{E_i(s)} = \frac{s+\frac{1}{T_c}}{s+\frac{1}{T_c\left(\frac{R_2}{R_1+R_2}\right)}}$$

$$\boxed{\text{Let } \alpha = \frac{R_2}{R_1+R_2}}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s+\frac{1}{T_c}}{s+\frac{1}{\alpha T_c}}}$$

Bode plot for lead network:

$$G_c = \frac{s+\frac{1}{T_c}}{s+\frac{1}{\alpha T_c}}$$

$$G_c = \alpha\left[\frac{1+T_cs}{1+\alpha T_cs}\right]$$

$$\boxed{G(j\omega) = \alpha\left[\frac{1+j\omega T_c}{1+j\omega\alpha T_c}\right]}$$

$$\omega_{c_1} = \frac{1}{T_c} \quad \text{and} \quad \omega_{c_2} = \frac{1}{\alpha T_c}$$

$$\therefore \boxed{\omega_{c_1} < \omega_{c_2}}$$

$$M = 20\log\left(\sqrt{1+\omega^2 T_c^2}\right) + \left(-20\log\sqrt{1+\alpha^2\omega^2 T_c^2}\right)$$

$$= 10\log(1+\omega^2 T_c^2) - 10\log(1+\alpha^2\omega^2 T_c^2)$$

$$\boxed{M = 10\log\left(1+\omega^2 T_c^2\right) - 10\log\left(1+\alpha^2\omega^2 T_c^2\right)} \qquad \boxed{\omega_{c_2} = \frac{1}{\alpha T_c}}$$

Maximum magnitude is at ω_{c_2} i.e at $\omega = \omega_{c_2}$

$$M = 10\log\left(1+\omega_{c_2}^2 T_c^2\right) - 10\log\left(1+\alpha^2\omega_{c_2}^2 T_c^2\right)$$

$$= 10\log\left(\frac{1}{\alpha^2 T_c^2}T_c^2\right) - 10\log\left(\alpha^2 T_c^2\left(\frac{1}{\alpha^2 T_c^2}\right)\right)$$

$$= 10\log\left(\frac{1}{\alpha^2}\right) - 10\log(1)$$

$$= 10\log\left(\frac{1}{\alpha}\right)^2$$

$$\boxed{M = 20\log\left(\frac{1}{\alpha}\right)}$$

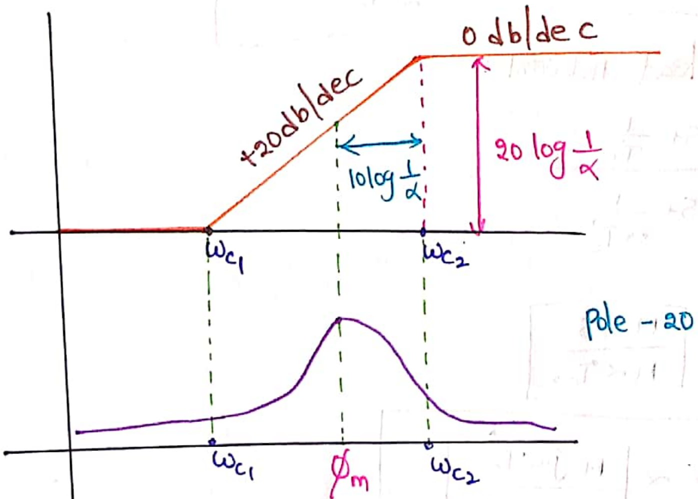
$$\phi = \tan^{-1}\left(\frac{\omega T_c}{1}\right) - \tan^{-1}\left(\frac{\alpha\omega T_c}{1}\right)$$

$$\phi = \tan^{-1}(\omega T_c) - \tan^{-1}(\alpha\omega T_c)$$

Derivate it and equate to zero

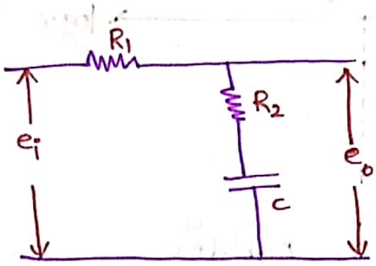
$$\omega_m = \frac{1}{\sqrt{\alpha}T}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right) - \tan^{-1}\left(\sqrt{\alpha}\right)}$$



$$\phi_m = \frac{1}{\sqrt{\alpha}T_c}$$

Lag Network:

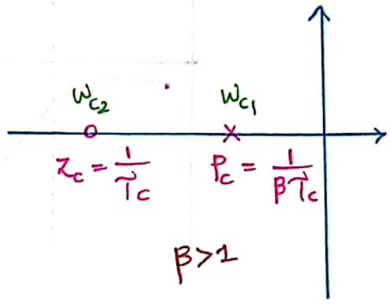


$$e_i = R_1 i + R_2 i + \frac{1}{C}\int i\,dt$$

$$\boxed{E_i(s) = R_1 I(s) + R_2 I(s) + \frac{1}{Cs} I(s)}$$

$$e_o = R_2 i + \frac{1}{C}\int i\,dt$$

$$\boxed{E_o(s) = R_2 I(s) + \frac{1}{Cs} I(s)}$$



$$\frac{E_o(s)}{E_i(s)} = \frac{\left(R_2 + \frac{1}{Cs}\right) I(s)}{\left(R_1 + R_2 + \frac{1}{Cs}\right) I(s)}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$

$$= \frac{1 + R_2 Cs}{(R_1 + R_2)Cs + 1}$$

$$= \frac{s + \frac{1}{R_2 C}}{\left(\frac{R_1 + R_2}{R_2}\right) s + \frac{1}{R_2 C}}$$

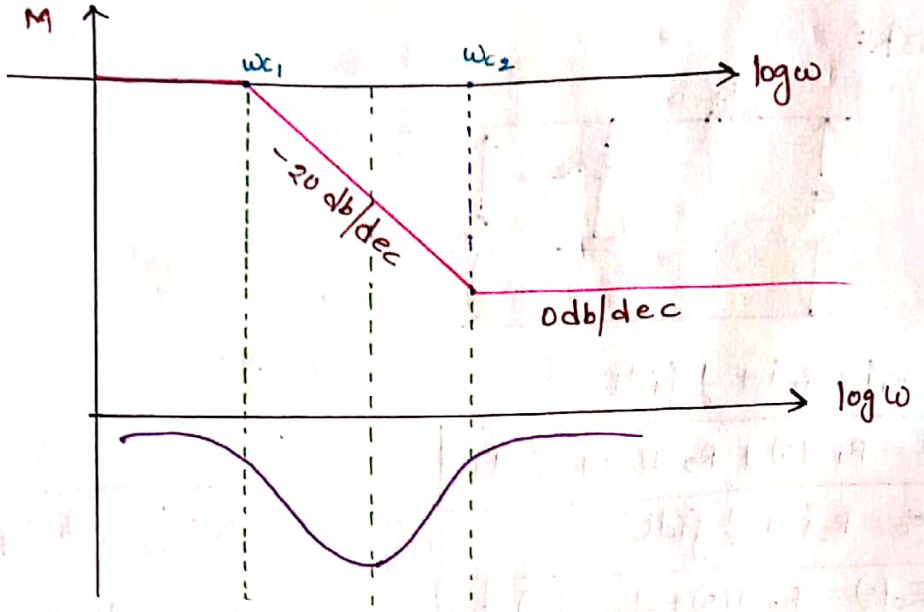
$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s + \frac{1}{R_2 C}}{\left(\frac{R_1 + R_2}{R_2}\right)\left[s + \frac{1}{\left(\frac{R_1 + R_2}{R_2}\right) R_2 C}\right]}}$$

Let $R_2 C = T_c$ and $\beta = \frac{R_1 + R_2}{R_2} > 1$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{s + \frac{1}{T_c}}{\beta\left(s + \frac{1}{\beta T_c}\right)}}$$

$$\phi = \underbrace{\tan^{-1}(\omega T_c)}_{\text{small}} - \underbrace{\tan^{-1}(\beta \omega T_c)}_{\text{High}}$$

$\boxed{\phi \text{ is } -\text{Ve}}$

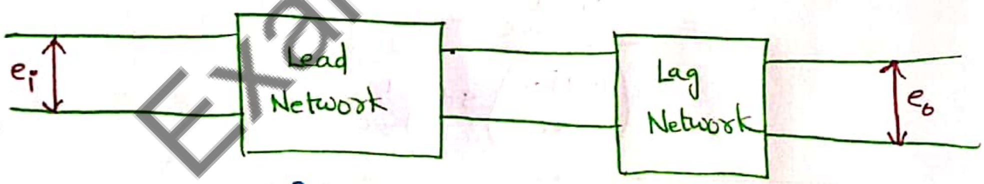


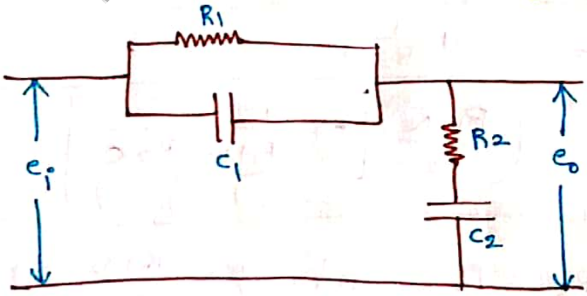
Lag-lead Network:

Lead Network, $G_c = \dfrac{s+\frac{1}{\tau_c}}{s+\frac{1}{\alpha\tau_c}}$ $\alpha<1$ τ_2

Lag Network, $G_c = \dfrac{s+\frac{1}{\tau_c}}{s+\frac{1}{\beta\tau_c}}$ $\beta>1$ τ_1

Lag-Lead Network, $G_c = \left[\left(\dfrac{s+\frac{1}{\tau_c}}{s+\frac{1}{\beta\tau_c}}\right)\left(\dfrac{s+\frac{1}{\tau_c}}{s+\frac{1}{\alpha\tau_c}}\right)\right]$





$R_1, C_1 \longrightarrow$ shows the lead effect

$R_2, C_2 \longrightarrow$ shows the lag effect.

$$Z = \frac{\frac{R_1}{C_1 s}}{R_1+\frac{1}{C_1 s}}$$

$$e_i = \left(\frac{R_1/C_1 s}{R_1+\frac{1}{C_1 s}}\right) i(t) + R_2 i(t) + \frac{1}{C_2}\int i(t) \quad \text{——} (1)$$

$$e_o = R_2 i(t) + \frac{1}{C_2}\int i(t) \quad \text{②}$$

From equation (1) and (2),

$$E_i(s) = \left[\frac{\frac{R_1}{C_1 s}}{R_1 + \frac{1}{C_1 s}} + R_2 + \frac{1}{C_2 s}\right] I(s)$$

$$E_o(s) = \left[R_2 + \frac{1}{C_2 s}\right] I(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\left[R_2 + \frac{1}{C_2 s}\right] I(s)}{\left(R_2 + \frac{1}{C_2 s} + \frac{\frac{R_1}{C_1 s}}{R_1 + \frac{1}{C_1 s}}\right) I(s)}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1 + R_2 C_2 s}{C_2 s}}{\left(\frac{R_2 C_2 s + 1}{C_2 s}\right) + \left(\frac{R_1}{R_1 C_1 s + 1}\right)}$$

$$= \frac{(1 + R_2 C_2 s)(1 + R_1 C_1 s)}{(1 + R_2 C_2 s)(1 + R_1 C_1 s) + R_1 C_2 s}$$

$$= \frac{R_1 C_1 R_2 C_2 \left(\frac{1}{R_1 C_1} + s\right)\left(\frac{1}{R_2 C_2} + s\right)}{s^2 (R_1 C_1 R_2 C_2) + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{\left(s + \frac{1}{R_2 C_2}\right)\left(s + \frac{1}{R_1 C_1}\right)}{s^2 + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1}\right) s + \frac{1}{R_1 C_1 R_2 C_2}}} \quad \text{③}$$

$$G_c = \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{1}{\beta T_1}\right)\left(s + \frac{1}{\alpha T_2}\right)}$$

$$\boxed{G_c = \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{s^2 + \left(\frac{1}{\alpha T_2} + \frac{1}{\beta T_1}\right) s + \frac{1}{T_1 T_2}}} \quad \text{④}$$

Comparing equation (3) and (4)

$T_1 = R_2 C_2$; $T_2 = R_1 C_1$

$\frac{1}{T_1 T_1} = \frac{1}{R_1 C_1 R_2 C_2}$; $T_1 T_2 = R_1 R_2 C_1 C_2$

$$\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} = \frac{1}{\alpha T_2} + \frac{1}{\beta T_1}$$

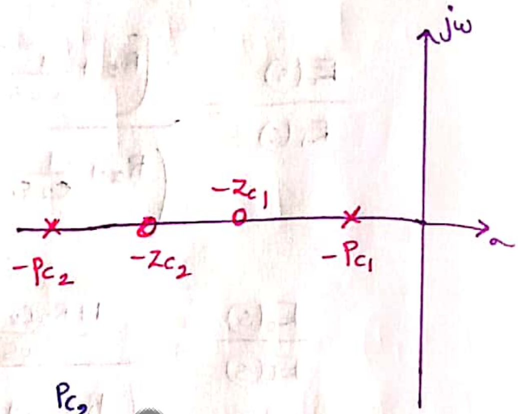
As $\alpha = \frac{1}{\beta}$

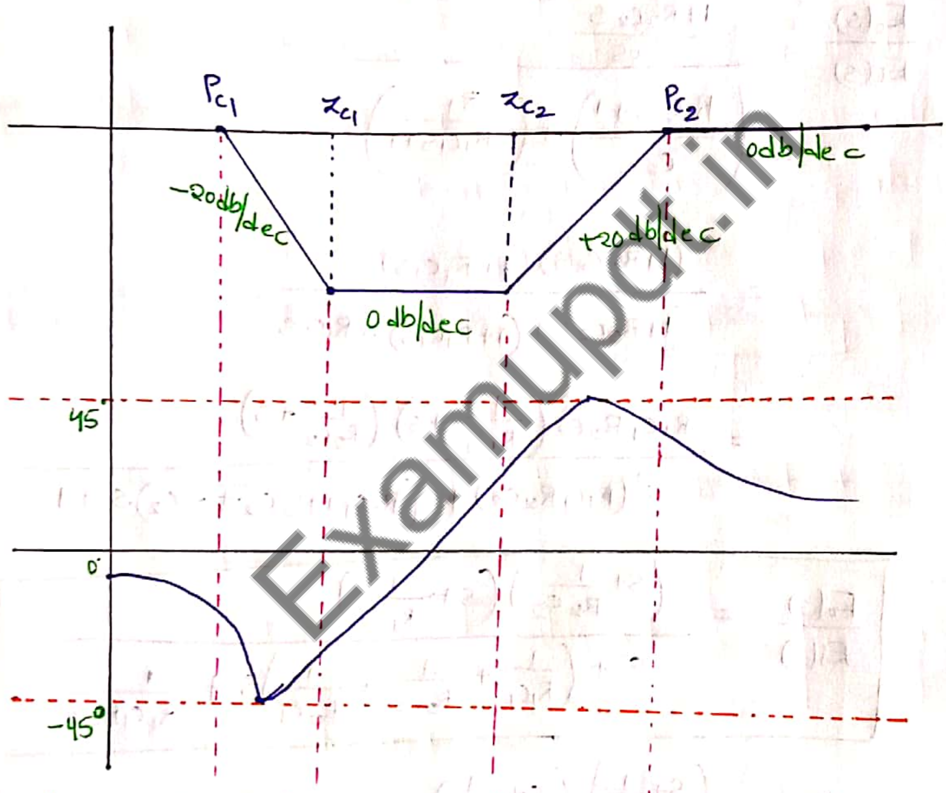
$$\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} = \frac{1}{\beta T_1} + \frac{\beta}{T_2}$$

$$G_c(j\omega) = \frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)\left(1+\frac{j\omega T_2}{\beta}\right)}$$

$-P_{c_1} = \frac{-1}{\beta T_1}$; $-P_{c_2} = \frac{-\beta}{T_2}$

$-Z_{c_1} = \frac{-1}{T_1}$; $-Z_{c_2} = \frac{-1}{T_2}$





Procedure for design of lead Compensator

Step-1: The system gain K is adjusted to meet the steady state error coefficients (K_p, K_v, K_a and e_{ss}) as given in problem.

Step-2: Plot the bode plot of the system with the desired gain 'K'. Determine the phase margin (ϕ_1) from the obtained phase plot. If the phase margin is satisfactory no Compensator is required. If the phase margin falls short (or) less than the ~~pha~~ desired additional phase lead ϕ_n must be

Provided by the lead Compensator at the gain cross-over frequency.

step-3: The maximum phase lead ϕ_m of a RC-lead network occurs at a frequency ω_m which is the geometric mean of the two corner frequencies ω_1 & ω_2 and it is desirable to have this phase lead at the gain cross-over frequency at uncompensated system.

Let $\omega_m = \omega_{gc}$ $\boxed{\sin\phi_m = \frac{1-\alpha}{1+\alpha}}$

Step-4: For the gain curve of the compensator lifts up the gain curve and the gain cross-over frequency gets changed.

Step-5: Adjust the gain K in a way such that the gain cross-over frequency is uneffected.

$\omega_m = \sqrt{\omega_1\omega_2}$

$\omega_1 = \omega_m\sqrt{\alpha}$

$\omega_2 = \frac{\omega_m}{\sqrt{\alpha}}$

$\omega_1 = \frac{1}{T_c}$

$\omega_2 = \frac{1}{\alpha T_c}$

$T_c = R_1C$

$\alpha = \frac{R_2}{R_1+R_2}$

(or)

Procedure for design of lead Compensator:

Step-1: Determine 'K' to meet the steady state requirements, specified by K_p, K_v, K_c (or) e_{ss}.

Step-2: For the open loop system plot the Bode plot with 'K' determined in step-1. Determine ω_{cg} (Gain-Cross over frequency) and ϕ_c (phase margin).

Step-3: Calculate the phase margin required from the Compensator ϕ_m.

Step-4: Determine 'α' such that required phase lead is the maximum phase lead of the compensator $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$

step-5: Determine $-10\log\frac{1}{\alpha}$, for the gain determine the frequency (ω_c) from magnitude plot.

step-6: Determine time constant T_c from ω_c (obtained in step-5) using formula $\boxed{\omega_c = \frac{1}{\sqrt{\alpha}\,\tau}}$ (or) $\boxed{\tau = \frac{1}{\omega_c\sqrt{\alpha}}}$

Step-7: Determine $K_c = \frac{K}{\alpha}$

Step-8: Determine the transfer function of Compensator

$$\boxed{G_c = \frac{K_c \alpha (1+\tau s)}{(1+\alpha \tau s)}}$$

Transfer function of overall system = $G_c(s) \times G(s)$

Step-9: Plot bode plot of Compensated system $[(G_c(s) \times G(s)]$ and ensure the specifications are met.

Step-10: Realize lead compensator with physical Components using the relations $\boxed{\alpha = \frac{R_2}{R_1+R_2}}$ and $\boxed{\tau = R_1 C}$

Assume R_1 (or) R_2 and Calculate remaining passive elements

Procedure to design a lag compensator:

Step-1: Determine 'K' from steady state error specifications (K_p, K_v, K_a (or) e_{ss})

Step-2: Plot the bode plot of uncompensated system with the value of K determined $KG(s)$. Determine ω_{cg} & phase margin.

Step-3: Find the phase lag required from the Compensator ϕ_m.

Step-4: The required ϕ_m is the maximum phase lag from Compensator. Now determine β. $\boxed{\sin\phi_m = \frac{1-\beta}{1+\beta}}$

Step-5: Find gain $10 \log\left(\frac{1}{\beta}\right)$. Find ω_c Such that at ω_c gain is $10 \log\left(\frac{1}{\beta}\right)$

Step-6: Find τ from obtained value of ω_c, $\boxed{\omega_c = \frac{1}{\sqrt{\beta}\,\tau}}$

Step-7: Find K_c, Such that $\boxed{K = K_c \beta}$

Step-8: Obtain the transfer function of Compensator as

$$\boxed{G_c(s) = \frac{K_c \beta (1+\tau s)}{(1+\beta \tau s)}}$$

Step-9: Plot the bode plot of Compensated system $G_c(s)\,G(s)$ and ensure if required specifications are met.

Step-10: Determine the passive Components R_1, R_2 & C using the relations, $\boxed{\tau = R_2 C}$; $\boxed{\beta = \frac{R_1+R_2}{R_2}}$