# Unit-5 : State Variable Analysis and Concept of state Variables

## Limitations of Transfer Function Analysis:

- Applicable only to SISO system
- For MIMO System - multiple transfer function
- As complexity increase process becomes tadious
- Applicable to time invariant systems
- zero initial Conditions.

Analysis done in frequency domain (s is Complex frequency).

## State space Analysis:

Tries to overcome majority of the limitations.

Space ⟶ mathematically set theory.

## state of a system:

The state of a dynamic system is the smallest set of Variables such that the knowledge of these Variables at $t=t_0$ together with knowledge of inputs for $t>t_0$, Completely determine the behaviour of the system.

## State Variables:

- The state variables the minimal set of Variables, as defined in state Which Completely define the behaviour of system for $t>t_0$
- Not necessarily physical variables and may not be measurable (or) observable.

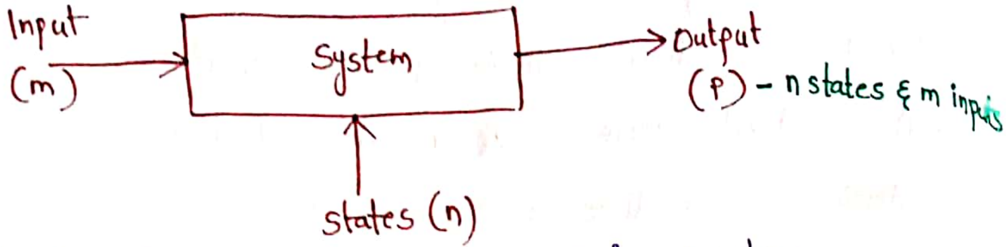
## State Vector:

If 'n' state Variables Completely define the behaviour of a system then these n Variables considered as the components of a vector $x(t)$ arranged in a Column matrix is called state Vector.

## State space:

The n dimensional space whose Coordinate axis consists of $x_1$ axis, $x_2$ axis, $x_3$ axis ....... $x_n$ axis where $x_1, x_2, x_3 \dots x_n$ are the n-state Variable is called the state space.

Variables ⟶ Input (m inputs) – m input Variables
output (p outputs) – p output Variables
state (n states) – n state variables




P output depends on 'n' states and 'm' inputs.

n states depends of before Current Instant of and m Inputs.

## State Space Equations:

In state space analysis, 3 types of Variables – Input, output and 4-state Variables. The state space representation of the system should define the output variables in terms of state variables and input variables. Similarly the state variables defined interms of states and inputs. Together all these equations form the state space equations.

Dynamic System – memory – memorize $t > t_0$

## Integrators:

Control system serves as a memory device. The outputs of such Integrator's Considered as state Variables.


$$\int \frac{dx}{dt} dt = x.$$

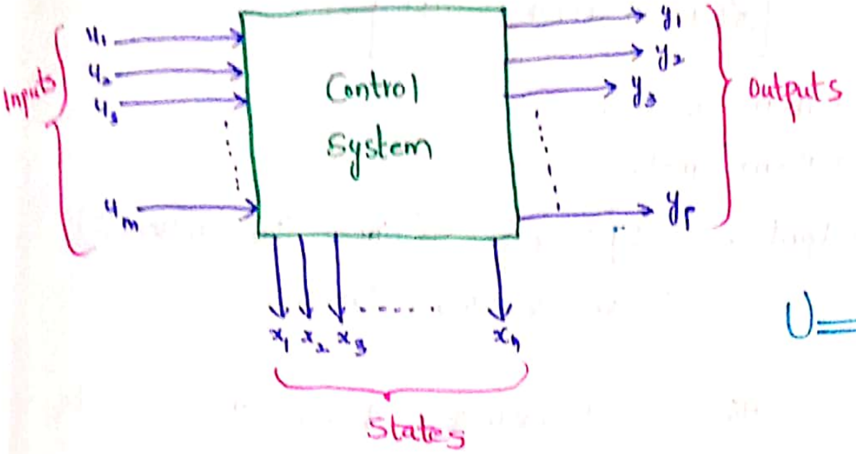
The input of state space model will be a first order differential equation.

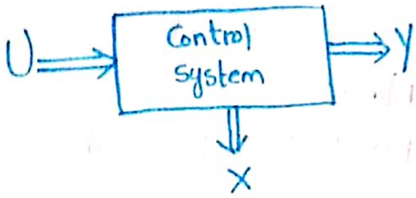
$t_1, t_0$ (time) $\qquad x_1(t_1), x_1(t_0)$

$$\boxed{\frac{dx_1}{dt} = \frac{x_1(t_1) - x_1(t_0)}{t_1 - t_0}}$$

$t_2, t_1$ (time) $\qquad x(t_2), x(t_1)$

$$\boxed{\frac{dx}{dt} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}}$$




Outputs ⇒ n states and m inputs

States → n states and m inputs.

n-state represented by n-first order differential equations.

$$\dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2, x_3 \dots x_n, u_1, u_2, \dots u_m)$$

$$\vdots$$

$$\dot{x}_n = \frac{dx_n}{dt} = f_n(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m)$$

$\dot{x}_i = f(x(t), u(t))$ → Linear Time Invariant

$\dot{x}_i = f(x(t), u(t), t)$ → Time Varying System.

State Model of Linear Time Invariant System.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{2n}u_n$$

As difficult to deal with them in equations, we use them in matrices

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{12} & b_{22} & \dots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{U}$$

State equation : $\boxed{x'(t) = A\,x(t) + B\,u(t)}$

$A \rightarrow n \times n$ matrix

$B \rightarrow n \times m$ matrix

A and B Constant $\longrightarrow$ LTI system (Linear Time Invariant)

$A(t)$ , $B(t)$ $\longrightarrow$ Time varying System.

Note:

For LTI System, the coefficients of matrices A & B are constants.
If A & B matrices are functions of time (t), then the system is time varying System.

## Output Variables:

$$y_1(t) = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$

$$y_2(t) = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$

$$\vdots$$

$$y_p(t) = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

In terms of matrices,

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{U}$$

Output equation,

$$Y = CX + DU$$

$C \rightarrow p \times n$ matrix

$D \rightarrow p \times m$ matrix

| | | |
|---|---|---|
| State Equation | – | $X' = AX + BU$ |
| Output Equation | – | $Y = CX + DU$ |

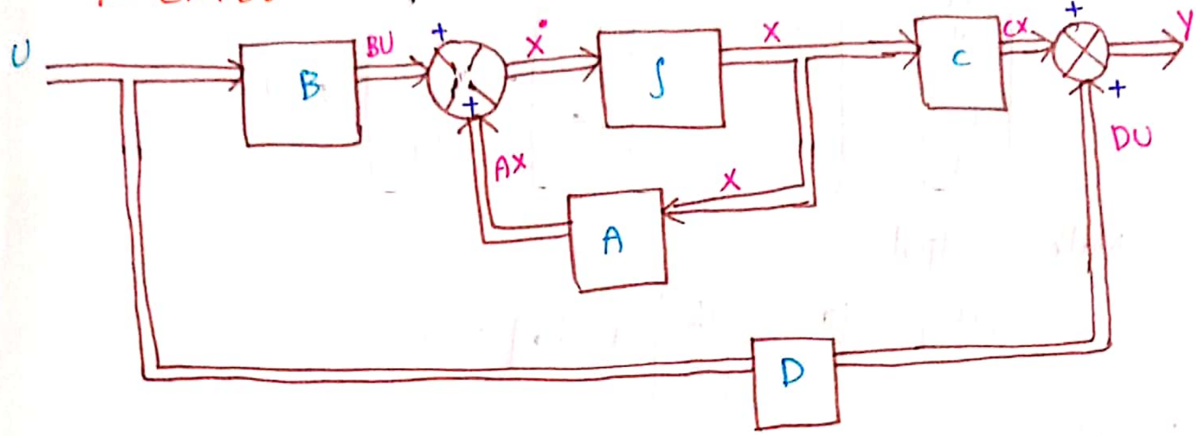
State space model of the System.

State Space model of the system

## Block Diagram for state space model of system

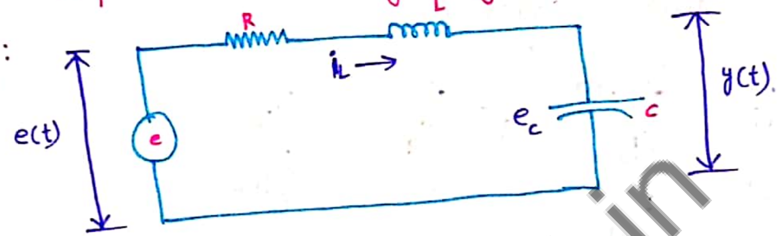
$\dot{X} = AX + BU$

$Y = CX + DU$




## State Space representation using physical Value:

**Example-1:**




Input - e(t) $\quad m = 1$

Output - y(t) $\quad p = 1$

State Variables $\quad n = 2 \; (i_L, V_c)$

Energy storage current in inductor and voltage across capacitor.

First order differential equations

$$\boxed{\begin{aligned} V_L &= L\frac{di}{dL} \\ I_c &= C\frac{dV_c}{dt} \end{aligned}}$$

All capacitor loop, All inductor cutset

Loop equation,

$$R\, i_L(t) + L\frac{di}{dt} + e_c(t) = e(t)$$

$$\boxed{\frac{di_L}{dt} = -\frac{R}{L}\, i_L(t) - \frac{1}{L}\, e_c(t) + \frac{1}{L}\, e(t)} \quad \text{—— (1)}$$

$$C\frac{de_c}{dt} = i_L$$

$$\boxed{\frac{de_c}{dt} = \frac{1}{c}\, i_L} \quad \text{—— (2)}$$

Output ; $\boxed{y(t) = e_c(t)}$ —— (3)

Matrix Input:

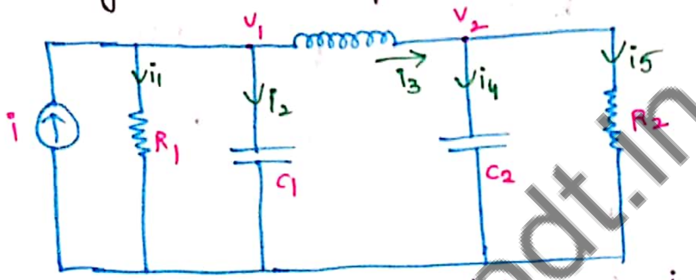
$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{de_c}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ e_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e(t)$$

Matrix output:

$$\begin{bmatrix} y(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ e_c \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} e(t)$$

## Example-2:

Write only state equation.




One input $i(t)$ ; $m = 1$. States - 3 $\quad i_3, V_1, V_2$.

KCL at node-1:

$$i = i_1 + i_2 + i_3$$

$$i = \frac{V_1}{R_1} + C_1 \frac{dV_1}{dt} + i_3$$

$$\boxed{\frac{dV_1}{dt} = \frac{-V_1}{R_1 C_1} - \frac{1}{C_1} i_3 + \frac{1}{C_1} i}$$ —— (1)

KCL at node-2:

$$i_3 = i_4 + i_5$$

$$i_3 = C_2 \frac{dV_2}{dt} + \frac{V_2}{R_2}$$

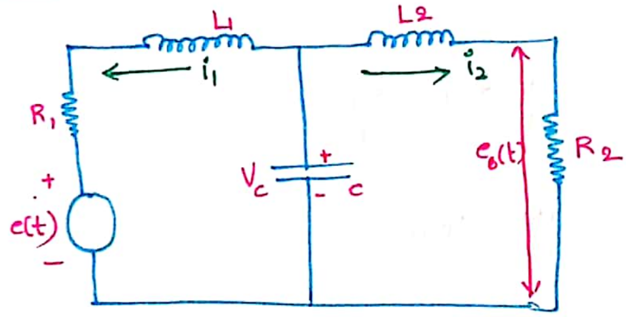
$$\boxed{\frac{dV_2}{dt} = \frac{-V_2}{R_2 C_2} + \frac{1}{C_2} i_3}$$ —— (2)

$$L\frac{di_3}{dt} + V_2 - V_1 = 0$$

$$\boxed{\frac{di_3}{dt} = \frac{1}{L}V_1 - \frac{1}{L}V_2}$$ —— (3)

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_1C_1} & 0 & \frac{-1}{C_1} \\ 0 & \frac{-1}{R_2C_2} & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \\ 0 \end{bmatrix} [i]$$

## Example-3 :




State Variables are $i_1, i_2, V_c$

KCL :

$$i_1 + i_2 + C\frac{dV_c}{dt} = 0$$

$$\boxed{\frac{dV_c}{dt} = \frac{-1}{C}i_1 - \frac{1}{C}i_2} \quad (3)$$

$$L_1\frac{di_1}{dt} + i_1R_1 + e(t) - V_c = 0$$

$$\boxed{\frac{di_1}{dt} = \frac{-R_1}{L_1}i + \frac{1}{L_1}V_c - \frac{1}{L_1}e(t)} \longrightarrow (1)$$

$$-V_c + L_2\frac{di_2}{dt} + R_2i_2 = 0$$

$$\boxed{\frac{di_2}{dt} = \frac{-R_2}{L_2}i_2 + \frac{1}{L_2}V_c} \longrightarrow (2)$$

output equation

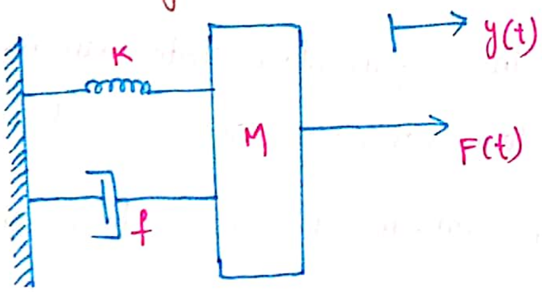
$$\boxed{e_o(t) = R_2i_2} \quad (4)$$

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dV_c}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ \frac{-1}{C} & \frac{-1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V_c \end{bmatrix} + \begin{bmatrix} \frac{-1}{L_1} \\ 0 \\ 0 \end{bmatrix} [e] \rightarrow \text{Input equation}$$

Output equation

$$[e_o] = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V_c \end{bmatrix} + [0][e]$$

## Example-4: Mechanical System



Input $= f(t)$ ; Output $= y(t)$

$$F = Ma = M\frac{d^2x}{dt^2} = M\frac{dv}{dt} \qquad (x, v)$$

$$F = Ky + f\frac{dy}{dt} + M\frac{d^2y}{dt^2}$$

Let $y = x_1$,

$$\frac{dy}{dt} = x_2 \quad \text{then}$$

$$\frac{dx_2}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$\frac{dy}{dt} = \frac{dx_1}{dt} \quad \text{then}$$

$$\dot{x}_1 = x_2$$

$$\boxed{\frac{dy}{dt} = \frac{dx_1}{dt} = \dot{x}_1 = x_2} \quad \text{—— (1)}$$

$$F = Kx_1 + fx_2 + M\frac{dx_2}{dt}$$

$$\boxed{\frac{dx_2}{dt} = \frac{-K}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}F} \quad \text{—— (2)}$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{M} & \frac{-f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F \qquad \text{Input equation}$$

$$[y] = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]F \qquad \text{output equation}$$

## Method – II

### State Space representation using phase Variables:

Let us have a transfer function.

Phase Variables $\longrightarrow$ System Variables $\longrightarrow$ output.

**Note:**

Phase Variables are those particular state Variables obtained from one of the system Variables and its derivatives.

* Normally the system Variable is chosen to be the system output

* Phase Variables are not physical quantities which cannot be measured and are easy to obtain state model from transfer function.

Differential form

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y + a_n y = b_0 u^m + b_1 u^{m-1} + \cdots + b_{m-1} u + b_m u$$

$(m=n)$

$$\text{Transfer function (T.F)} = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

Transfer function is of $n^{th}$ order then it has 'n' state Variables

$$(x_1, x_2, x_3, \ldots\ldots, x_n)$$

Let

$$x_1 = y$$
$$x_2 = \dot{y} = \dot{x}_1$$
$$x_3 = \ddot{y} = \dot{x}_2$$
$$x_4 = \dddot{y} = \dot{x}_3$$
$$\vdots$$
$$x_{n-1} = y^{n-2} = \dot{x}_{n-2}$$
$$x_n = y^{n-1} = \dot{x}_{n-1}$$
$$\dot{x}_n = y^n$$

All Coefficients of bj (except) $b_n = 0$ and $b_n = b$

(Transfer function),

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

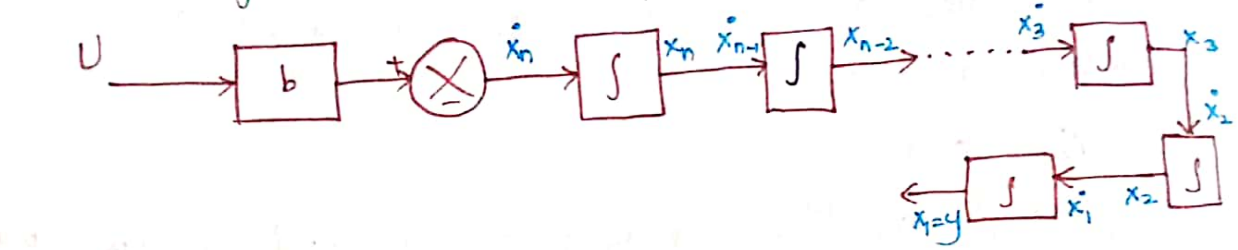
$$\dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots\cdots + a_{n-1} x_2 + a_n x_1 = bu$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots\cdots - a_2 x_{n-1} - a_1 x_n + bu$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & & a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ b \end{bmatrix} u$$

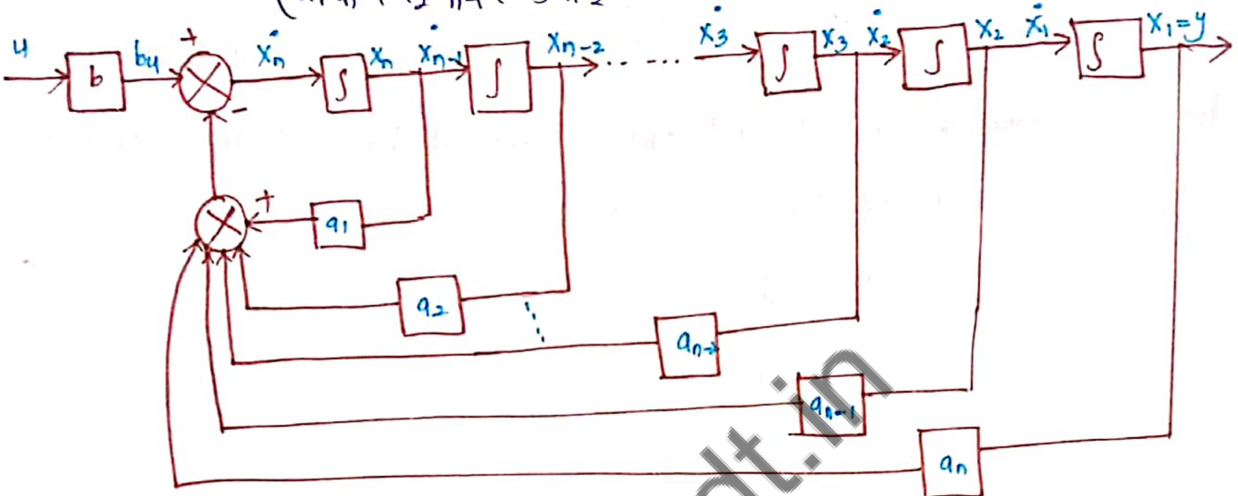
Bush Companion form

Block diagram for Bush companion form:



Above is the equation line.

$$\dot{x}_n = bu - (a_1 x_n + a_2 x_{n-1} + a_3 x_{n-2} + \cdots + a_{n-1} x_2 + a_n x_1)$$



Considering system with both poles and zeros.

Consider a third order system,

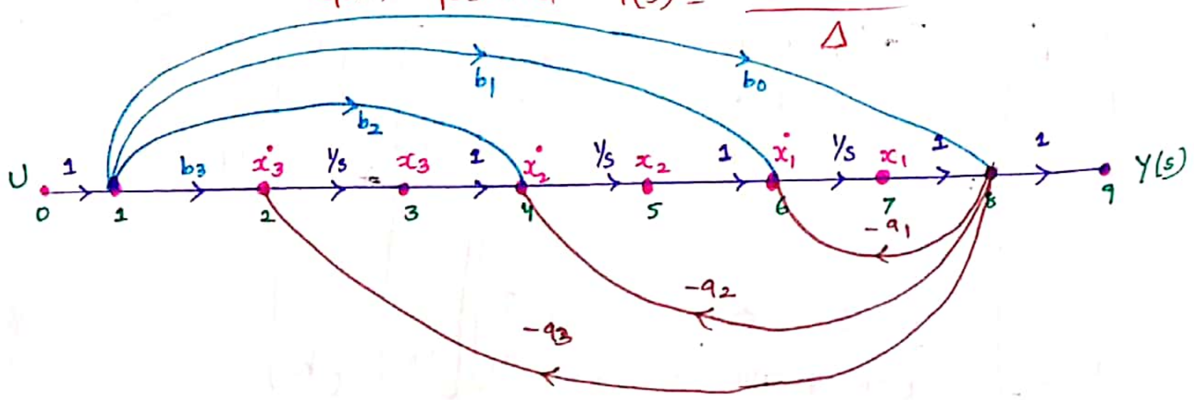
$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Third-order system has 3-state Variables has $x_1, x_2, x_3$

Divide both numerator and denominator with $s^3$.

$$T(s) = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right)}$$

Masons Gain formula $T(s) = \frac{\Sigma P_K \Delta K}{\Delta}$



The expressions are,

$$y = x_1 + b_0 u$$

$$\dot{x}_1 = x_2 + (-a_1)(x_1 + b_0 u) + b_1 u = -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u$$

$$\dot{x}_2 = x_3 + b_2 u - a_2(x_1 + b_0 u) = -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$$

$$\dot{x}_3 = b_3 u + (-a_3)(x_1 + b_0 u) = -a_3 x_1 + (b_3 - a_3 b_0) u$$

In matrices form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 & -a_1 b_0 \\ b_2 & -a_2 b_0 \\ b_3 & -a_3 b_0 \end{bmatrix} u$$ Input equation

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$ output equation

It is also a canonical form.

Canonical Form (or) Jordan block:

Eigen Values:

$$T(s) = \frac{c_1}{s-\lambda_1} + \frac{c_2}{s-\lambda_2} + \cdots + \frac{c_n}{s-\lambda_n}$$

Eigen Values are the roots of characteristic equation

In matrices form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$ Input equation

$$[y] = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [b_0] u$$ Output equation

If all roots are distinct

If all roots are distinct but $\lambda_1 = \lambda_2 = \lambda$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$[y] = [c_1 \; c_2 \; c_3 \; \cdots \; c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [b_0] u$$

By making Jordan block into upper diagonal matrix

- If three roots are same then make 3x3 matrix into upper diagonal matrix.

## State Model Equations:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

As $\dot{x} \rightarrow n \times 1$ ; $X \rightarrow n \times 1$ ; $u \rightarrow m \times 1$ ; $y \rightarrow p \times 1$

Then A [square matrix] $\longrightarrow n \times n$

B $\longrightarrow n \times m$

C $\longrightarrow p \times n$

D $\longrightarrow p \times m$

$\dot{x} \rightarrow$ state equation

$y \rightarrow$ output equation

Where A is state matrix (or) System matrix

B is Input matrix

C is output matrix

D is Transmission matrix.

## Transfer function from state model

$$\dot{x} = AX + BU$$
$$Y = CX + DU.$$

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$sX(s) - AX(s) = x(0) + BU(s)$$
$$(sI - A)X(s) = x(0) + BU(s)$$

$$\boxed{X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}B\,U(s)}$$

$$Y(s) = C X(s) + D U(s)$$

$$= C\left[(sI-A)^{-1}x(0)\right] + C\left[(sI-A)^{-1}B\,U(s)\right] + D\,U(s)$$

$$\boxed{Y(s) = C(sI-A)^{-1}x(0) + \left[C(sI-A)^{-1}B + D\right]U(s)}$$

If zero Initial Conditions $x(0)=0$

$$Y(s) = \{C(sI-A)^{-1}B + D\}\,U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D}$$ (Transfer function)

1) Solve $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3+6s^2+10s+5}$

Sol:

$x_1 = y$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -5 & -10 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$ Input equation

$$[y] = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]\,u$$ output equation

2) Solve $\dfrac{Y(s)}{U(s)} = \dfrac{5}{s^3+6s+7}$

Sol

$b_0 = 5,\quad a_1 = 0,\quad a_2 = 6,\quad a_3 = 7$

$x_1 = y$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -7x_1 - 6x_2 + 5u$

$(s^3+6s+7)\,Y(s) = 5\,U(s)$

$\dddot{y} + 6\dot{y} + 7y = 5u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$ Input equation

$$[y] = [1 \; 0 \; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$ Output equation

3) Solve $\dfrac{Y(s)}{U(s)} = \dfrac{s^2+3s+4}{s^3+2s^2+3s+2}$

Sol:

$a_0 = 1$ $b_0 = 0$

$a_1 = 2$ $b_1 = 1$

$a_2 = 3$ $b_2 = 3$

$a_3 = 2$ $b_3 = 4$

$$(s^3+2s^2+3s+2)\, Y(s) = (s^2+3s+4)\, U(s)$$

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 2y = \ddot{u} + 3\dot{u} + 4u$$

$\beta_0 = b_0 = 0$

$\beta_1 = b_1 - a_1 b_0 = 1 - 2(0) = 1$

$\beta_2 = b_2 - a_2 b_0 = 3 - 3(0) = 3$

$\beta_3 = b_3 - a_3 b_0 = 4 - 2(0) = 4$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} u$$ Input equation

$$[y] = [1 \; 0 \; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$ Output equation

4 Solve $\dfrac{Y(s)}{U(s)} = \dfrac{2s^3+7s^2+12s+8}{s^3+6s^2+11s+9}$

Sol.

$a_0 = 1$ $b_0 = 2$

$a_1 = 6$ $b_1 = 7$

$a_2 = 11$ $b_2 = 12$

$a_3 = 9$ $b_3 = 8$

$$(s^3+6s^2+11s+9)Y(s) = (2s^3+7s^2+12s+8)U(s)$$

$$\dddot{y}+6\ddot{y}+11\dot{y}+9y = 2\dddot{u}+7\ddot{u}+12\dot{u}+8u$$

$$\beta_0 = b_0 = 2$$

$$\beta_1 = b_1 - a_1 b_0 = 7-6(2) = -5$$

$$\beta_2 = b_2 - a_2 b_0 = 12-(11)(2) = -10$$

$$\beta_3 = b_3 - a_3 b_0 = 8-9(2) = -10$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ x_3\end{bmatrix} = \begin{bmatrix}-6 & 1 & 0\\ -11 & 0 & 1\\ -9 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}-5\\ -10\\ -10\end{bmatrix}\begin{bmatrix}u\end{bmatrix}$$ , Input equation

Output equation.

$$\begin{bmatrix}y\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}2\end{bmatrix}\begin{bmatrix}u\end{bmatrix}$$

Extended matrix:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}-a_1 & 1 & 0 & \cdots & 0 & 0\\ -a_2 & 0 & 1 & \cdots & 0 & 0\\ -a_3 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & & \\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1\\ -a_n & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}b_1-a_1b_0\\ b_2-a_2b_0\\ b_3-a_3b_0\\ \vdots\\ b_{n-1}-a_{n-1}b_0\\ b_n-a_nb_0\end{bmatrix} U(s)$$

$$\begin{bmatrix}y\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}b_0\end{bmatrix}u$$

# Decomposition of Transfer function

There are two methods in this decomposition. They are

1) Direct Decomposition
   (i) First Companion Form (Controllable canonical form)
   (ii) Second Companion Form (observable Canonical form.
2) Indirect decomposition.

In In-direct decomposition there are more ② types. They are:

- Cascade decomposition
- parallel decomposition

We need to develop first and second forms with Controllable & observability

## 1) First Companion form:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n}$$

Divide the numerator and denominator with $s^n$

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + \dots + b_{n-1} s^{-(n-1)} + b_n s^{-n}}{1 + a_1 s^{-1} + a_2 s^{-2} + \dots + a_{n-1} s^{-(n-1)} + a_n s^{-n}}$$

Step-1: Express the Transfer function in -ve powers of s.

Step-2: Multiply and divide $\frac{Y(s)}{U(s)}$ with a dummy Variable $X(s)$.

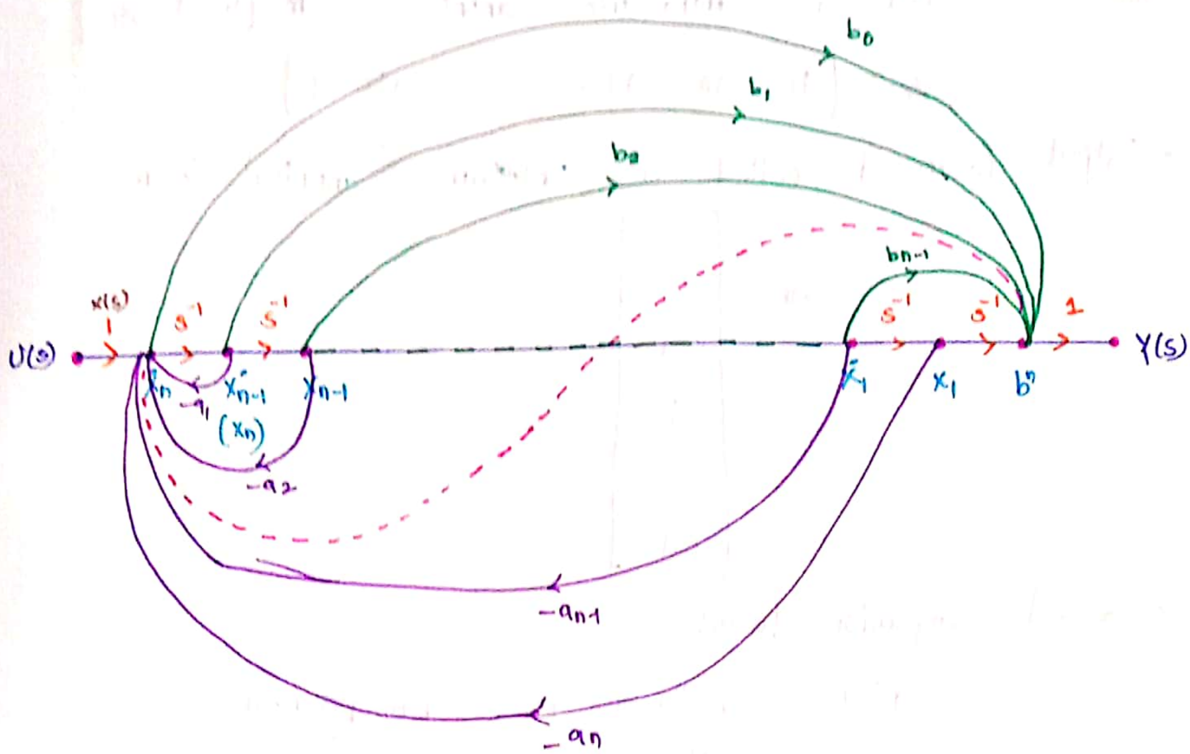
$$Y(s) = \left[b_0 + b_1 s^{-1} + b_2 s^{-2} + \dots + b_n s^{-n}\right] X(s)$$

$$U(s) = \left[1 + a_1 s^{-1} + a_2 s^{-2} + \dots + a_n s^{-n}\right] X(s)$$

Step-3: Write Cause and effect relations.

$$X(s) = U(s) - a_1 s^{-1} X(s) - a_2 s^{-2} X(s) - \dots - a_n s^{-n} X(s)$$

Next we need to draw Single flow graph for those expressions.




$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\vdots$

$$\boxed{\dot{x}_n = -a_1 x_n - a_2 x_{n-1} - a_3 x_{n-2} - \cdots - a_{n-1} x_2 - a_n + u(s)}$$

$$y = b_n x_1 + b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1} x_2$$

$$y = b_0(-a_1 x_n - a_2 x_2 - \cdots - a_{n-1} x_2 - a_n x_1) + b_n x_1 + b_{n-1} x_2 + \cdots + b_2 x_{n-1} + b_1 x_n$$

$$\boxed{y = (b_n - a_n b_0) x_1 + (b_{n-1} - a_{n-1} b_0) x_2 + \cdots + (b_2 - a_2 b_0) x_{n-1} + (b_1 - a_1 b_0) x_n}$$

In matrices form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u]$$

Input

$$[y] = \begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \cdots & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [b_0] u$$

output

• This is called as Controllable Canonical form [Input form]

$$Q_c = \begin{bmatrix} B & AB & A^2B & \cdots\cdots & A^{n-1} & B \end{bmatrix}$$

• Output form is called as observable Canonical form

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

## 2) Second Companion Form:

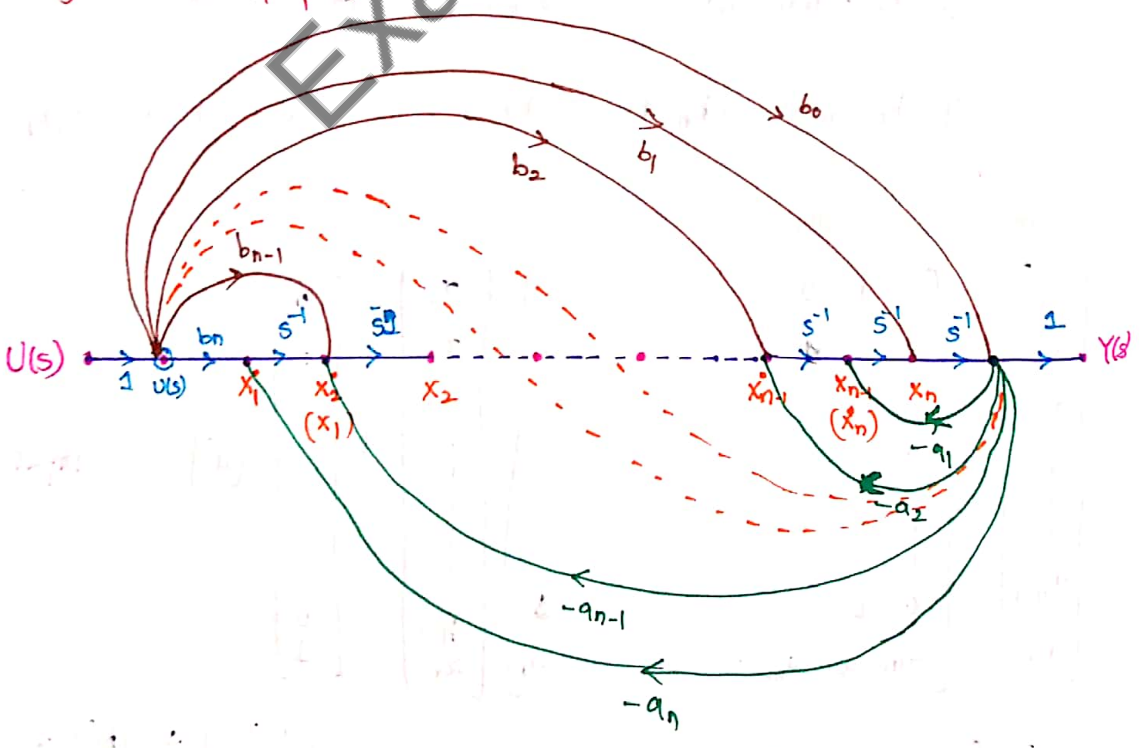
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{-1} + \cdots + b_{n-1} s^{-(n-1)} + b_n s^{-n}}{1 + a_1 s^{-1} + \cdots + a_{n-1} s^{-(n-1)} + a_n s^{-n}}$$

$$\left[1 + a_1 s^{-1} + \cdots + a_{n-1} s^{-(n-1)} + a_n s^{-n}\right] Y(s) = \left[b_0 + b_1 s^{-1} + \cdots + b_{n-1} s^{-(n-1)} + b_n s^{-n}\right] U(s)$$

$$Y(s) = \left[-a_1 s^{-1} + \cdots - a_{n-1} s^{-(n-1)} - a_n s^{-n}\right] Y(s) + \left[b_0 + b_1 s^{-1} + \cdots + b_{n-1} s^{-(n-1)} + b_n s^{-n}\right] U(s)$$

Signal Flow Graph,



$$\dot{x}_1 = b_n u(s) - a_n(x_n + b_0 u) = -a_n x_n + (b_n - a_n b_0)u$$

$$\dot{x}_2 = -a_{n-1}(x_n + b_0 u) + b_{n-1}u + x_1 = -a_{n-1}x_n + (b_{n-1} - a_{n-1}b_0)u + x_1$$

$$\vdots$$

$$\dot{x}_{n-1} = -a_2 x_n + (b_2 - a_2 b_0)u + x_{n-2}$$

$$\dot{x}_n = -a_1 x_n - (b_1 - a_1 b_0)u + x_{n-1}$$

$$y(t) = x_n + b_0 u$$

In matrices form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$ Input

$$[y] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [b_0]u$$ Output

∴ Observable Canonical form

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

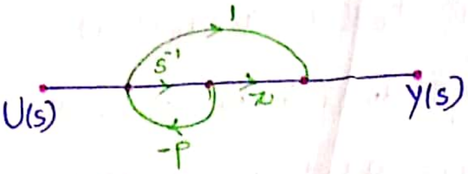
Contrallable canonical form

## 3) Series Decomposition (or) Cascade Decomposition

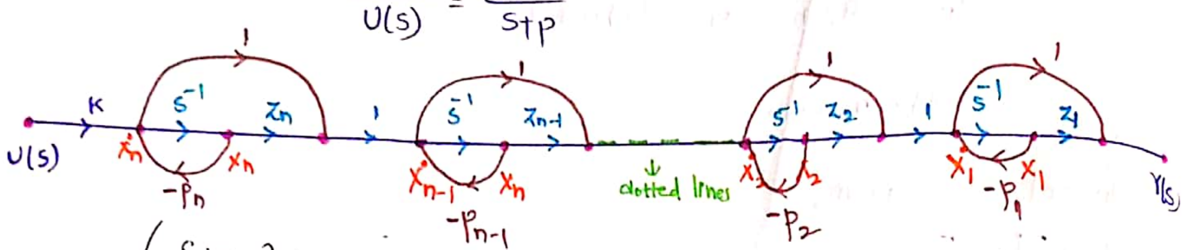
$$\frac{Y(s)}{U(s)} = \frac{K(s+z_1)(s+z_2)\cdots\cdots(s+z_{n-1})(s+z_n)}{(s+p_1)(s+p_2)\cdots\cdots(s+p_{n-1})(s+p_n)}$$

$$\frac{Y(s)}{U(s)} = K\left(\frac{s+z_1}{s+p_1}\right)\left(\frac{s+z_2}{s+p_2}\right)\cdots\cdots\left(\frac{s+z_{n-1}}{s+p_{n-1}}\right)\left(\frac{s+z_n}{s+p_n}\right)$$

$$\frac{S+z}{S+p}$$




$$\frac{Y(s)}{U(s)} = \frac{S+z}{S+p}$$




$$\left(\frac{S+z_n}{S+p_n}\right) \qquad \left(\frac{S+z_{n-1}}{S+p_{n-1}}\right) \qquad \left(\frac{S+z_2}{S+p_2}\right) \qquad \left(\frac{S+z_1}{S+p_1}\right)$$

$$\dot{x}_1 = -p_1 x_1 + (z_2-p_2)x_2 + (z_3-p_3)x_3 + \cdots + (z_n-p_n)x_n + kU$$

$$\dot{x}_2 = -p_2 x_2 + z_3 x_3 + \dot{x}_3 = -p_2x_2 + (z_3-p_3)x_3 + (z_4-p_4)x_4 + \cdots + (z_n-p_n)x_n + kU$$

$$\vdots$$

$$\dot{x}_{n-1} = -p_{n-1}x_{n-1} + z_n x_n + \dot{x}_n = -p_{n-1}x_{n-1} + (z_n-p_n)x_n + kU$$

$$\dot{x}_n = -p_n x_n + kU$$

$$y(t) = z_1x_1 + \dot{x}_1 = (z_1-p_1)x_1 + (z_2-p_2)x_2 + (z_3-p_3)x_3 + \cdots + (z_n-p_n)x_n + kU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & (z_2-p_2) & (z_3-p_3) & \cdots & (z_n-p_n) \\ 0 & -p_2 & (z_3-p_3) & \cdots & (z_n-p_n) \\ 0 & 0 & -p_3 & \cdots & (z_n-p_n) \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots -p_{n-1} & (z_n-p_n) \\ 0 & 0 & 0 & \cdots 0 & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} K \\ K \\ K \\ \vdots \\ K \\ K \end{bmatrix} U$$

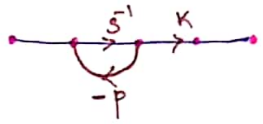
$$[y] = \begin{bmatrix} (z_1-p_1) & (z_2-p_2) & (z_3-p_3) & \cdots & (z_{n-1}-p_{n-1}) & [z_n-p_n] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [K]U$$

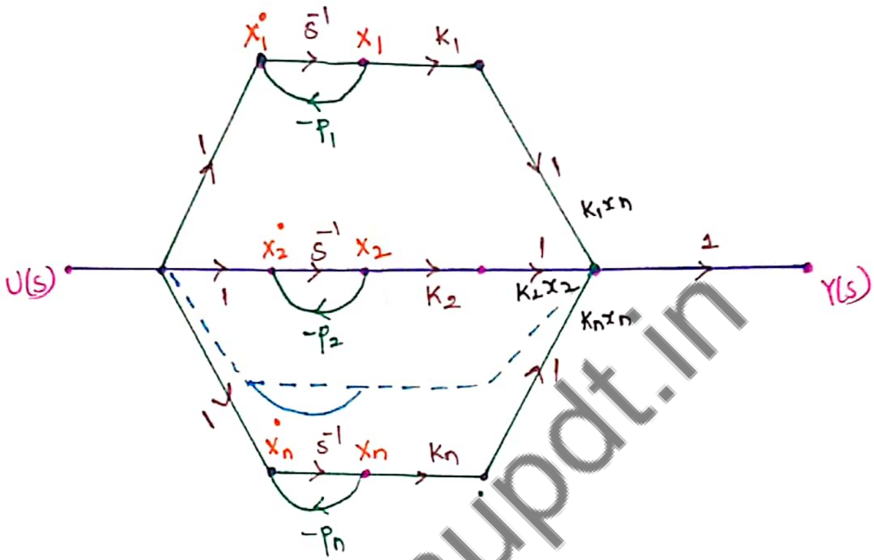
## 4) Parallel Decomposition:

$$\frac{Y(s)}{U(s)} = \frac{Q(s)}{(s+P_1)\ (s+P_2)\cdots\cdots(s+P_{n-1})(s+P_n)}$$

$$\frac{Y(s)}{U(s)} = \frac{K_1}{s+P_1} + \frac{K_2}{s+P_2} + \frac{K_3}{s+P_3} + \cdots\cdots + \frac{K_{n-1}}{s+P_{n-1}} + \frac{K_n}{s+P_n}$$

$$\frac{K}{s+P}$$




$$\dot{x}_1 = -P_1x_1 + U$$

$$\dot{x}_2 = -P_2x_2 + U$$

$$\vdots$$

$$\dot{x}_n = -P_nx_n + U$$

$$y(t) = K_1x_1 + K_2x_2 + \cdots\cdots + K_nx_n$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -P_1 & 0 & \cdots & 0 \\ 0 & -P_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & -P_n \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [U]$$

$$[y] = [K_1\ K_2\ \cdots\ K_n]\,x(t) + [0][U]$$

$A \longrightarrow$ diagonal matrix

It is called Diagonal Canonical form

## Solution of state Equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Complete Solution $y(t)$ ; $Y = CX + DU$

if homogeneous $u(t) = 0$

if non-homogeneous $u(t) \neq 0$

## Homogeneous State Equation [input $u(t) = 0$]

$$\dot{x}(t) = Ax(t)$$

Where $x(t) \rightarrow$ vector

$A \rightarrow$ square matrix

Consider a scalar,

Let $\frac{dx}{dt} = ax(t)$

$x(0) = x_0$

$$x(t) = \left[1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots\right] x_0$$

$$x(t) = e^{at} x_0 \qquad \left[\because e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots\right]$$

$$\therefore \boxed{x(t) = e^{at} x_0}$$

Extending to matrices (By analogy);

$$\frac{d}{dt}(x(t)) = Ax(t) \qquad \text{[Analogy = Similarly]}$$

$$x(t) = e^{At} x_0$$

$$\boxed{x(t) = e^{At} x_0}$$

Where $e^{At} \rightarrow$ matrix exponential

Where $e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$

$$x(t) = e^{At} x_0$$

$$\phi(t) = e^{At} \rightarrow \text{state Transition matrix (STM)}$$

- For any value of 't' the value of $x(t)$ given by the value of $x(0)$ is called state transition matrix $\phi(t)$.
- Initial time $(t=0)$ $t = t_0$, then STM $\phi(t - t_0) = e^{A(t-t_0)}$

Now Consider the non-homogeneous equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x}(t) - Ax(t) = Bu(t)$$

Premultiply both sides with $e^{-At}$

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At}Bu(t)$$

$$\boxed{\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}\left[\dot{x}(t) - Ax(t)\right]}$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t)$$

Integrating on both sides (from 0 to t)

$$\int \frac{d}{dt} e^{-At} x(t)\, d(t) = \int e^{-At} B u(t)$$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau} Bu(\tau)\, d\tau$$

Premultiply $e^{At}$ on both sides

$$x(t) = e^{At}x(0) + \int_0^t e^{At} e^{-A\tau} Bu(\tau)\, d\tau$$

$$= e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d(\tau)$$

$$\therefore \boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d(\tau)}$$

$e^{At} \rightarrow$ state Transition matrix.

Properties of STM [state Transition Matrix]:

1) $\phi(0) = I$

2) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$

3) $\phi(t_2-t_1) \cdot \phi(t_1-t_0) = \phi(t_2-t_0)$

$$\phi(t_2-t_1)\,\phi(t_1-t_0) = e^{A(t_2-t_1)}\, e^{A(t_1-t_0)}$$

$$= e^{At_2} \cdot e^{-At_1} \cdot e^{At_1} \cdot e^{-At_0}$$

$$= e^{At_2} \cdot e^{-At_0}$$

$$= e^{A(t_2-t_0)}$$

$$\phi(t_2-t_1)\,\phi(t_1-t_0) = \phi(t_2-t_0)$$

4) $[\phi(t)]^k = \phi(kt)$

$$[\phi(t)]^k = e^{At} e^{At} \dots \text{ k times}$$

$$= e^{kAt}$$

$$[\phi(t)]^k = \phi(kt)$$

5) $\phi(t_1+t_2) = \phi(t_1)\,\phi(t_2)$

## Computation of STM (State Transition Matrix) [matrix exponential] $e^{At}$.

1) Infinite Series Expansions
2) Laplace Transform method
3) Cayley Hamilton Theorem
4) Canonical Transformation method

### 1) Infinite Series Expansions:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

Taking a 2x2 matrix, $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} t + \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}\frac{t^2}{2} + \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}\frac{t^3}{6} + \dots$$

$$\boxed{e^{At} = \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{2t^3}{6} + \dots & t - t^2 + \frac{t^3}{2} + \dots \\ -t + t^2 + \frac{t^3}{2} + \dots & 1 - 2t + \frac{3t^2}{2} - \frac{2t^3}{3} + \dots \end{bmatrix}}$$

Homework:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}t + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\frac{t^3}{6} + \cdots$$

$$\boxed{e^{At} = \begin{bmatrix} 1+t+\frac{t^2}{2}+\frac{t^3}{6}+\cdots & 1+t^3+\frac{t^3}{2}+\cdots \\ 0 & 1+t+\frac{t^2}{2}+\frac{t^3}{6}+\cdots \end{bmatrix}}$$

## Laplace Transformation Method:

$$\dot{x}(t) = A\,x(t)$$

$$S\,X(s) - x(0) = A\,X(s)$$

$$S\,X(s) - A\,X(s) = x(0)$$

$$[SI - A]\,X(s) = X(0)$$

Premultiply both sides with $[SI-A]^{-1}$

$$X(s) = [SI-A]^{-1}x(0)$$

$$\boxed{x(t) = L^{-1}\{(SI-A)^{-1}\}\,x(0)}$$

**Scalar Case:**

$$L^{-1}\{(s-a)^{-1}\} = L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$x(t) = e^{At}x(0)$$

$$\therefore \boxed{e^{At} = L^{-1}\{(SI-A)^{-1}\}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 1 & S+2 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{S(S+2)-(-1)\times 1}\begin{bmatrix} S+2 & 1 \\ -1 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S+2}{(S+1)^2} & \frac{1}{(S+1)^2} \\ \frac{-1}{(S+1)^2} & \frac{S}{(S+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(S+1)^2}+\frac{1}{S+1} & \frac{1}{(S+1)^2} \\ \frac{-1}{(S+1)^2} & \frac{1}{(S+1)^2}+\frac{1}{S+1} \end{bmatrix}$$

Taking inverse laplace transform,

$$\phi(At) = \begin{bmatrix} te^{-t}+e^{-t} & te^{-t} \\ -te^{-t} & -te^{-t}+e^{-t} \end{bmatrix}$$

$$\boxed{\phi(At) = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}}$$

2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

By using laplac Transformation method:

$$e^{At} = L^{-1}\{(SI-A)^{-1}\}$$

$$SI-A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & S-1 \\ 0 & S-1 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{(S-1)^2-0}\begin{bmatrix} S-1 & 1 \\ 0 & S-1 \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{1}{S-1} & \frac{1}{(S-1)^2} \\ 0 & \frac{1}{S-1} \end{bmatrix}$$

$$e^{At} = L^{-1}\{(SI-A)^{-1}\}$$

$$\boxed{e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}}$$

## Cayley Hamilton Theorem:

- Every square matrix satisfies its own characteristic equation.
- Eigen values are the roots of characteristic equation.

characteristic equation $|\lambda I - A| = 0$

$A$ is a square matrix

## Matrix polynomial:

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{n-1} A^{n-1}$$

Where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are constants.

this constant is obtained from eignen values.

$A_{(n\times n)} \rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values.

Where $\lambda_1, \lambda_2, \ldots \lambda_n$ are disctinct.

Define: $f(\lambda_i) = \alpha_0 + \alpha_1 \lambda_i + \alpha_2 \lambda_i^2 + \cdots + \alpha_{n-1} \lambda_i^{n-1}$

for $i = 1, 2, 3, \ldots n$

$\alpha_0, \alpha_1, \alpha_2 \ldots \alpha_n$ are n-coefficients

$\lambda_1, \lambda_2, \ldots \lambda_n$ have n-equations.

## Formal procedure:

Step-1: Find the eigen values of $A$ $[|\lambda I - A| = 0]$

Step-2: If all eigen values are distinct obtain the n-equations of $f(\lambda_i)$ then solve to determine the coefficients of $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$.

- If any of the roots are repeated, write one independent equation by substitution and the remaining equations by differentiating the independent equation.

Step-3: Substitute the coefficients $\alpha_0, \alpha_1, \alpha_2 \ldots \alpha_{n-1}$ obtained in step 2 in the matrix polynomial

1) Solve the $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of SIM in Cayley Hamilton Theorem.

Sol: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|\lambda I - A| = 0$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-1)^2$$

To find the eigen values, $|\lambda I - A| = 0$

$$(\lambda-1)^2 = 0$$

$\boxed{\lambda_1 = +1}$ and $\boxed{\lambda_2 = 1}$

Matrix polynomial; $e^{At} = f(A) = \alpha_0 I + \alpha_1 A$

Consider $\lambda_1 = 1$ $\quad e^{\lambda t} = f(\lambda) = \alpha_0 \cdot 1 + \alpha_1 \cdot \lambda$ [scalar case with $\lambda$]

$$\alpha_0 + \alpha_1 \lambda = e^{\lambda t} \qquad \boxed{e^t = \alpha_0 + \alpha_1} \text{ — (1)} \quad [\because \lambda = 1]$$

To get 2nd equation differentiate both sides w.r.t to $\lambda$.

$$\frac{d}{d\lambda}(\alpha_0 + \alpha_1 \lambda) = \frac{d}{d\lambda} e^{\lambda t}$$

$$\alpha_1 = t e^{\lambda t}$$

$\lambda = 1$; $\quad \boxed{\alpha_1 = te^t}$ — (2)

From equation (1),

$$\alpha_0 + \alpha_1 = e^t$$

$$\alpha_0 + te^t = e^t$$

$$\boxed{\alpha_0 = e^t - te^t}$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$= (e^t - te^t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}}$$

2) Compute STM $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in Cayley Hamilton method.

Sol:

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

$$|\lambda I - A| = \lambda(\lambda+3) + 2 = \lambda^3 + 3\lambda + 2$$

To find the eigen values, $|\lambda I - A| = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\boxed{\lambda_1 = -1}, \quad \boxed{\lambda_2 = -2}$$

Matrix polynomial, $e^{At} = f(A) = \alpha_0 I + \alpha_1 A$

Scalar case with $\lambda$; $e^{\lambda t} = f(\lambda) = \alpha_0 I + \alpha_1 \lambda$

$\lambda_1 = 1$, $e^{-1t} = f(-1) = \alpha_0 - \alpha_1$ —— ①

$\lambda_2 = -2$, $e^{-2 \cdot t} = f(-2) = \alpha_0 - 2\alpha_1$ —— ②

Solve equation ① & ②,

$$\alpha_0 - \alpha_1 = e^{-t}$$

$$\alpha_0 - 2\alpha_1 = e^{-2t}$$

$$\alpha_1 = e^{-t} - e^{-2t}$$

Substitute $\alpha_1$ in equation (1),

$$\alpha_0 - e^{-t} + e^{-2t} = e^{-t}$$

$$\boxed{\alpha_0 = 2e^{-t} - e^{-2t}}$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{At} = (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2(e^{-t} - e^{-2t}) & 2e^{-t} - e^{-2t} - 3(e^{-t} - e^{-2t}) \end{bmatrix}$$

$$\therefore\ \boxed{e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & -e^{-t} + 2e^{-2t} \end{bmatrix}}$$

**Homework:**

Compute STM $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ in Cayley Hamilton method.

**Sol:**

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix}$$

$$|\lambda I - A| = \lambda(\lambda+2)+1 = \lambda^2+2\lambda+1$$

To find the eigen values, $|\lambda I - A| = 0$

$$|\lambda I - A| = (\lambda+1)^2 = 0$$

$$\boxed{\lambda_1 = -1} \qquad \boxed{\lambda_2 = -1}$$

Matrix polynomial, $e^{At} = f(A) = \alpha_0 I + \alpha_1 A$

Consider Scalar case with $\lambda$, $e^{\lambda t} = f(\lambda) = \alpha_0 \cdot 1 + \alpha_1 \lambda$

Let $\lambda_1 = -1$; $e^{-1 \cdot t} = f(-1) = \alpha_0 - \alpha_1$

$$\boxed{e^{-t} = \alpha_0 - \alpha_1} \quad \text{——— (1)}$$

To get 2nd equation differentiate both sides with respect to $\lambda$

$$\frac{d}{d\lambda}(\alpha_0 + \alpha_1 \lambda) = \frac{d}{d\lambda}(e^{\lambda t})$$

$$\alpha_1 = t e^{\lambda t}$$

$\lambda = -1$; $\boxed{\alpha_1 = t e^{-t}}$ ——— (2)

From equation (1),

$$e^{-t} = \alpha_0 - t e^{-t}$$

$$\boxed{\alpha_0 = e^{-t} + t e^{-t}}$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{At} = \left[e^{-t} + t e^{-t}\right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t e^{-t} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}+te^{-t}-2(te^{-t}) \end{bmatrix}$$

$$\therefore \boxed{e^{At} = \begin{bmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix}}$$

## Canonical Transformation method:

Using model matrix,

$A$ into diagonal form.

$$\begin{tikzcd} A \arrow[r] & \hat{A} \arrow[d] \\ e^{At} \arrow[r] & e^{\hat{A}t} \end{tikzcd}$$

$\dot{x}(t) = Ax(t) \qquad [\because x(t) = e^{At}x(0)]$

$x(t) = Mz(t)$

$$\boxed{\frac{d}{dt}(Mz(t)) = A\cdot Mz(t)}$$

$M\dot{z}(t) = A\cdot M\cdot z(t)$

Premultiply both sides with $M^{-1}$.

$M^{-1}\cdot M\cdot \dot{z}(t) = M^{-1}AMz(t) \qquad [M^{-1}M = 1]$

$\dot{z}(t) = \hat{A}z(t)$

$\hat{A} \rightarrow$ diagonal matrix.

$$\boxed{z(t) = e^{\hat{A}t}z(0)}$$

This is Solution of above equation

$z(t) = M^{-1}x(t)$

$z(t) = e^{\hat{A}t}z(0)$

$M^{-1}x(t) = e^{\hat{A}t}M^{-1}x(0)$

$x(t) = Me^{\hat{A}t}M^{-1}x(0) \qquad [\because x(t) = e^{At}x(0)]$

$Me^{\hat{A}t}M^{-1}x(0) = e^{At}x(0)$

$$\therefore \boxed{e^{At} = Me^{\hat{A}t}M^{-1}}$$

Procedure:

$$\boxed{M \Rightarrow \hat{A} \Rightarrow e^{\hat{A}t} \Rightarrow e^{At}}$$

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{\lambda_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$A \rightarrow \hat{A} \rightarrow e^{\hat{A}t} \rightarrow e^{At}$$

$$A \rightarrow \hat{A} = M^{-1}AM \longrightarrow e^{At} = M e^{\hat{A}t} M^{-1}$$

Jordan block $\rightarrow$ Jacobian matrix

Consider $\lambda_1$ is repeating (3) times.

Jordan block

$$J = \left[\begin{array}{ccc|cccc} \lambda_1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & \lambda_1 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & \lambda_1 & 0 & \cdots & & 0 \\ \hline 0 & 0 & 0 & \lambda_4 & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & & \lambda_n \end{array}\right]$$

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & \frac{t^3}{3!}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

If we get repeated roots, it is called as Jordon blocks. It is called as Jacobian Matrix 'J'.

$$2\times2 \rightarrow e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix}$$

1) Compute SIM $\begin{bmatrix} 1 & 1 \\ 0 & +1 \end{bmatrix}$

Sol:

$$\lambda_1 = \lambda_2 = +1 \quad , \quad M = [M_1 \vdots M_2]$$

Eigen Vector $[\lambda_1 I - A]$ ;

Cofactors Transpose

$$[\lambda I - A] = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$c_{11} = 0$ ; $c_{12} = 0$ , $c_{21} = 1$ , $c_{22} = 0$

Cofactor Transpose

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Eigen Vector,

$$M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda_2 (\lambda_2 = \lambda_1) \longrightarrow$ We can't get directly eigen values

$$[\lambda_1 I - A] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 - 1 & -1 \\ 0 & \lambda_1 - 1 \end{bmatrix}$$

Transpose matrix. $\begin{bmatrix} \lambda_1 - 1 & 0 \\ -1 & \lambda_1 - 1 \end{bmatrix}$

$$M_2 = \begin{bmatrix} \frac{d}{d\lambda}(c_{21}) \\ \frac{d}{d\lambda}(c_{22}) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$ and

$$\hat{A} = M^{-1} A M = A \ ;$$

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J \text{ (Jordon block)}$$

$$\therefore e^{\hat{A}t} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \quad \text{as} \quad \hat{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = M e^{\hat{A}t} \cdot M^{-1} = e^{\hat{A}t}$$

$$\boxed{e^{\hat{A}t} = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}}$$

2) Compute SIM $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in Canonical Transformation model.

Sol:

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

To find eigen values $|\lambda I - A| = 0$

$$|\lambda I - A| = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\boxed{\lambda_1 = -1} \qquad \boxed{\lambda_2 = -2}$$

$$M_1 = [\lambda_1 I - A] = -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$c_{11} = 2$, $c_{12} = -2$, $c_{21} = 1$, $c_{22} = -1$

Cofactors $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

$$M_1 = \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M_2 = [\lambda_2 I - A] = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

$c_{11} = 1$, $c_{12} = -2$, $c_{21} = 1$, $c_{22} = -2$

$$M_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boxed{M = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}$$

$$M^{-1} = \frac{1}{-2+1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\hat{A} = M^{-1}AM = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{\hat{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$e^{At} = M e^{\hat{A}t} M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}}$$

Transfer Function from state model:

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

Taking laplace transform

$$Y(s) = C[sI - A]^{-1} X(0) + \{C(sI-A)^{-1}B + D\} U(s)$$

1)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Sol:
$$(sI-A)^{-1} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s[s(s+3) - (-1)(2)] - (-1)[0+1] + 0} \begin{bmatrix} (s+1)(s+2) & (s+3) & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{s^3+3s^2+2s+1}\begin{bmatrix}(s+1)(s+2) & (s+3) & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}$$

$T(s) = C[SI-A]^{-1}B + D$ $\qquad \boxed{D=0}$

$$T(s) = \frac{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}}{s^3+3s^2+2s+1}\begin{bmatrix}(s+1)(s+2) & (s+3) & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{s^3+3s^2+2s+1}\begin{bmatrix}(s+1)(s+2) & (s+3) & 1\\ -s & -(2s+1) & s^2\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{s^3+3s^2+2s+1}\begin{bmatrix}s+3 & 1\\ -(2s+1) & s^2\end{bmatrix}$$

$$T(s) = \begin{bmatrix}\frac{s+3}{s^3+3s^2+2s+1} & \frac{1}{s^3+3s^2+2s+1}\\ \frac{-(2s+1)}{s^3+3s^2+2s+1} & \frac{s^2}{s^3+3s^2+2s+1}\end{bmatrix} \quad \begin{bmatrix}\frac{Y_1}{U_1} & \frac{Y_1}{U_2}\\ \frac{Y_2}{U_1} & \frac{Y_2}{U_2}\end{bmatrix}$$

Homework:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$[Y] = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

Sol: $(SI-A)^{-1} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}$

$$(SI-A)^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

Transfer function;

$$T(s) = C(SI-A)^{-1}B + D$$

Since, $D=0$

$$T(s) = \frac{\begin{bmatrix} 1 & 2 \end{bmatrix}}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \frac{\begin{bmatrix} s+3-4 & 1+2s \end{bmatrix}}{s^2+3s+2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} s-1 & 2s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{T(s) = \left[\frac{2s+1}{s^2+3s+2}\right]}$$

## Controllability and Observability

### Controllability of Linear Systems

- At $t = t_0$ we know $x(t_0)$ then by giving a known Input $u(t)$ then we can transfer

For finite time: $x(t_0) \longrightarrow x(t)$ desired state.

- A system is said to be Completely State Controllable at time $t_0$, if it is possible by means of an un Constrained Control Vector $u(t)$ to transfer the system from initial state $x(t_0)$ to any other desired state in a finite interval of time

Completely Controllable $\longrightarrow [x_1, x_2, x_3, \dots x_n]$ (State Variables)

### Completely Controllable :

- A process is said to be Completely Controllable, if all the state Variables of the process can be Controlled to reach a certain desired Value in a finite time period by an Unconstrained Control Input $u(t)$
- If any of the states is independent of the input then there would be no way this state Variable can be driven to the desired value and that particular state is uncontrollable.

## Observability

- The system is said to be observable at time '$t_0$' if with the knowledge of $x(t_0)$, it is possible to determine the state of the system from the observation of the output over a finite interval of time

- Kalman introduced the concept of controllability and observability.
- Kalman's test is used to test for controllability and observability

## Test for Controllability

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) \rightarrow n\times 1 \text{ state vector.}$$
$$y(t) = Cx(t) + Du(t)$$

- For the system to be completely state controllable, it is necessary and sufficient that the $n \times nm$ matrix given by,

$$Q_c = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \cdots \,\vdots\, A^{n-1}B]$$ should be of rank '$n$'

- Order of B is $n\times m$
- $Q_c$ has n-rows and n-columns of m.



## Output Controllability:

- In practical design of a control system we may want to control the output of the system rather than the state of the system. The system described by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is completely output controllable if it is possible to construct a control vector $u(t)$ that will transfer the system from initial output $y(t_0)$ to a desired output $y(t)$ in a finite interval of time.

- The condition for output controllability is the '$p \times (n+1)m$' matrix $Q = [CB \,\vdots\, CAB \,\vdots\, CA^2B \,\vdots\, \cdots \,\vdots\, CA^{n-1}B \; D]$ is of rank '$p$' where $p$ is the number of outputs.

1. $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$ check whether it is state Controllable.

Sol:

$$Q_c = [B \quad AB \quad A^2B]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \qquad [A \cdot AB]$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

$|Q_c|$ = det of $Q_c$ = $0 - 0 + 1(0-1) = -1 \neq 0$

So, the matrix is not Singular

$\therefore \boxed{\text{Rank}(Q_c) = 3}$

$\therefore$ The System is Completely state Controllable.

2. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Sol.

$$Q_c = [B \quad AB]$$

$$AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$|Q_c|$ = det of $Q_c = 0$

Rank of $Q_c \neq 2$

$\therefore$ So, the System is not Completely state Controllable.

3. $A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}$  $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Check if it is State Controllable and Output Controllable.

Sol:

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix}$$

$$|Q_c| = -11 \neq 0$$

∴ It is Completely state Controllable

$$Q = \begin{bmatrix} CB & CAB \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$$

$$CAB = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3 & -5 \end{bmatrix}$$

∴ Rank = 1

∴ Output Controllable matrix.



Observability Completely Observable:

- An LIT system described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$Y(t) = Cx(t) + Du(t)$$

is said to be Completely observable, if every state $x(t_0)$ can be determined by the observation of $y(t)$. The Condition of observability depends on the matrices 'A' & 'C'

- Kalman's test for observability states that the system is Completely observable if the rank of the observability matrix $Q_0$ is 'n'

## Kalman's test for Observability:

Kalman's test for Observability states that the system is completely observable if and only if (iff) the rank of the Observability matrix $Q_0$ is 'n'.

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C^T & A^TC^T & (A^2)^TC^T & \cdots & (A^{n-1})^TC^T \end{bmatrix}$$

1. $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Sol:

$C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A^T = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^TC^T = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} C^T & A^TC^T \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank $= 1 \neq 2$.

Not completely observable.



## Duality property (or) principle:

- If the pair A and B is controllable, it implies the pair $A^T$, $B^T$ is observable.
- If the pair A and C is controllable, it implies the pair $A^T$, $B^T$ is controllable.

$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ Controllable

$\Rightarrow \begin{bmatrix} C^T & A^TC^T & (A^2)^TC^T & \cdots & (A^T)^{n-1}C \end{bmatrix}$

$\Rightarrow \begin{bmatrix} B^T & A^TB^T & (A^T)^2B^T & \cdots & (A^T)^{n-1}B^T \end{bmatrix}$ observable

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## Homework:

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$. Check for state Controllability, Output Controllability and observability

Sol:

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 3 & 10 \end{bmatrix}$$

$$|Q_c| = 0 - 1(0-0) + 4(0-0) = 0$$ [singular matrix]

Rank $\neq 3$

It is not state Controllable.

$$Q = \begin{bmatrix} CB & CAB & CA^2B \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$CAB = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$CA^2B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}$$

Rank $= 1$

It is Completely output Controllability.

$$C^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A^TC^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & -14 & 10 \end{bmatrix}.$$

$$(A^2)^T = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -14 \\ 4 & 0 & 10 \end{bmatrix}$$

$$(A^2)^T C^T = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -14 \\ 4 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$|Q_0| = 1(12-1) - 1(4-0) + 2(1-0) = 9 \neq 0$$

So, the matrix is non-singular

$\therefore$ Rank = 3

It is Completely observable.

2. $A = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Check for state Controllability, Output Controllability and Observability.

Sol:

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$|Q_c| = 0 - 1 = -1$$

Rank = 2

It is Completely State Controllable

$$Q = \begin{bmatrix} CB & CAB \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$CAB = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Rank = 1

It is completely output controllability.

$$Q_0 = \begin{bmatrix} C^T & A^T C^T \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_0| = 1 - 0 = 1$$

$\therefore$ Rank = 2

It is completely observable.

3. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$ check observability.

Sol: $Q_0 = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ +36 & 60 & 25 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \quad (A^2)^T = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix}$$

$$(A^2)^T C^T = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$|Q_0| = 0$$

$\therefore$ Rank $\neq 3$

$\therefore$ It is not Completely observability.